Problems: Divergence Theorem

Let $S_1$ be the part of the paraboloid $z = 1 - x^2 - y^2$ which is above the $xy$-plane and $S_2$ be the unit disk in the $xy$-plane. Use the divergence theorem to find the flux of $F$ upward through $S_1$, where $F = (yz, xz, xy)$.

**Answer:** Write $F = Mi + Nj +Pk$, where $M = yz$, $N = xz$, and $P = xy$. Then

$$\text{div}F = M_x + N_y + P_z = 0.$$

The divergence theorem says: flux $= \iiint_{S_1+S_2} F \cdot \mathbf{n} \, dS = \iiint_D \text{div}F \, dV = \iiint_D 0 \, dV = 0$

$\Rightarrow \iiint_{S_1} F \cdot \mathbf{n} \, dS + \iiint_{S_2} F \cdot \mathbf{n} \, dS = 0 \Rightarrow \iiint_{S_1} F \cdot \mathbf{n} \, dS = -\iiint_{S_2} F \cdot \mathbf{n} \, dS.$

Therefore to find what we want we only need to compute the flux through $S_2$.

But $S_2$ is in the $xy$-plane, so $dS = dx \, dy$, $\mathbf{n} = -\mathbf{k}$ $\Rightarrow F \cdot \mathbf{n} \, dS = -xy \, dx \, dy$ on $S_2$.

Since $S_2$ is the unit disk, symmetry gives

$$\iiint_{S_2} -xy \, dx \, dy = 0 \Rightarrow \iiint_{S_1} F \cdot \mathbf{n} \, dS = -\iiint_{S_2} F \cdot \mathbf{n} \, dS = 0.$$
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