V13.3 Stokes' Theorem

3. Proof of Stokes' Theorem.

We will prove Stokes' theorem for a vector field of the form $P(x,y,z)\,\mathbf{k}$. That is, we will show, with the usual notations,

$$\oint_C P(x,y,z)\,dz \;=\; \iint_S \text{curl }(P\,\mathbf{k}\,)\cdot\mathbf{n}\,dS\;. \tag{3}$$

We assume S is given as the graph of $z = f(x,y)$ over a region R of the xy-plane; we let C be the boundary of S, and C' the boundary of R. We take $\mathbf{n}$ on S to be pointing generally upwards, so that $|\mathbf{n}\cdot\mathbf{k}\,| = \mathbf{n}\cdot\mathbf{k}$.

To prove (3), we turn the left side into a line integral around C', and the right side into a double integral over R, both in the xy-plane. Then we show that these two integrals are equal by Green's theorem.

To calculate the line integrals around C and C', we parametrize these curves. Let

$$C':\;\; x = x(t),\;\; y = y(t), \qquad\qquad\qquad t_0 \le t \le t_1$$

be a parametrization of the curve C' in the xy-plane; then

$$C:\;\; x = x(t),\;\; y = y(t),\;\; z = f(x(t), y(t)), \quad t_0 \le t \le t_1$$

gives a corresponding parametrization of the space curve C lying over it, since C lies on the surface $z = f(x,y)$.

Attacking the line integral first, we claim that

$$\oint_C P(x,y,z)\,dz \;=\; \oint_{C'} P(x,y,f(x,y))(f_x dx + f_y dy)\;. \tag{4}$$

This looks reasonable purely formally, since we get the right side by substituting into the left side the expressions for z and dz in terms of x and y: $z = f(x,y)$, $\quad dz = f_x dx + f_y dy$. To justify it more carefully, we use the parametrizations given above for C and C' to calculate the line integrals.

$$\begin{aligned}
\oint_C P(x,y,z)\,dz \;&=\; \int_{t_0}^{t_1} \big(P(x(t),y(t),z(t)\big)\,\frac{dz}{dt}\,dt \\
&=\; \int_{t_0}^{t_1} \big(P(x(t),y(t),z(t)\big)\,(f_x\frac{dx}{dt} + f_y\frac{dy}{dt})\,dt,\ \text{by the chain rule} \\
&=\; \oint_{C'} P(x,y,f(x,y))(f_x dx + f_y dy), \quad \text{the right side of (4).}
\end{aligned}$$

We now calculate the surface integral on the right side of (3), using x and y as the variables. In the calculation, we must distinguish carefully between such expressions as $P_1(x,y,f)$ and $\frac{\partial}{\partial x}P(x,y,f)$. The first of these means: calculate the partial derivative with respect to the first variable x, treating x,y,z as independent; then substitute $f(x,y)$ for z. The second means: calculate the partial with respect to x, after making the substitution $z = f(x,y)$; the answer is

$$\frac{\partial}{\partial x} P(x,y,f) \;=\; P_1(x,y,f) + P_3(x,y,f)\, f_x \;.$$

(We use P_1 rather than P_x since the latter would be ambiguous — when you use numerical subscripts, everyone understands that the variables are being treated as independent.)

With this out of the way, the calculation of the surface integral is routine, using the standard procedure of an integral over a surface having the form $z = f(x,y)$ given in Section V9. We get

$$\begin{aligned}
d\mathbf{S} \;&=\; (-f_x\,\mathbf{i} - f_y\,\mathbf{j} + \mathbf{k}\,)\,dx\,dy, \qquad\qquad \text{by V9, (13);}\\
\text{curl}\ (P(x,y,z)\ \mathbf{k}\,) \;&=\; \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ 0 & 0 & P \end{vmatrix} \;=\; P_2(x,y,z)\,\mathbf{i} - P_1(x,y,z)\,\mathbf{j}\\
\iint_S \text{curl}\ (P(x,y,z)\ \mathbf{k}) \cdot d\mathbf{S} \;&=\; \iint_S \big(-P_2(x,y,z)f_x + P_1(x,y,z)f_y\big)\,dx\,dy\\
&=\; \iint_R \big(-P_2(x,y,f)f_x + P_1(x,y,f)f_y\big)\,dx\,dy \qquad (5)
\end{aligned}$$

We have now turned the line integral into an integral around C' and the surface integral into a double integral over R. As the final step, we show that the right sides of (4) and (5) are equal by using Green's theorem:

$$\oint_{C'} U\,dx + V\,dy \;=\; \iint_R (V_x - U_y)\,dx\,dy \;.$$

(We have to state it using U and V rather than M and N, or P and Q, since in three-space we have been using these letters for the components of the general three-dimensional field $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$.) To substitute into the two sides of Green's theorem, we need four functions:

$$\begin{aligned}
V \;&=\; P(x,y,f(x,y))f_y, \qquad \text{so} \qquad V_x \;=\; (P_1 + P_3 f_x)f_y + P(x,y,f)f_{yx}\\
U \;&=\; P(x,y,f(x,y))f_x, \qquad \text{so} \qquad U_y \;=\; (P_2 + P_3 f_y)f_x + P(x,y,f)f_{xy}
\end{aligned}$$

Therefore, since $f_{xy} = f_{yx}$, four terms cancel, and the right side of Green's theorem becomes

$$V_x - U_y \;=\; P_1(x,y,f)f_y - P_2(x,y,f)f_x,$$

which is precisely the integrand on the right side of (5). This completes the proof of Stokes' theorem when $\mathbf{F} = P(x,y,z)\,\mathbf{k}$.

In the same way, if $\mathbf{F} = M(x,y,z)\,\mathbf{i}$ and the surface is $x = g(y,z)$, we can reduce Stokes' theorem to Green's theorem in the yz-plane.

If $\mathbf{F} = N(x,y,z)\,\mathbf{j}$ and $y = h(x,z)$ is the surface, we can reduce Stokes' theorem to Green's theorem in the xz-plane.

Since a general field $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ can be viewed as a sum of three fields, each of a special type for which Stokes' theorem is proved, we can add up the three Stokes' theorem equations of the form (3) to get Stokes' theorem for a general vector field.

A difficulty arises if the surface cannot be projected in a 1-1 way onto each of three coordinate planes in turn, so as to express it in the three forms needed above:

$$z = f(x, y), \qquad x = g(y, z), \qquad y = h(x, z)\ .$$

In this case, it can usually be divided up into smaller pieces which can be so expressed (if some of these are parallel to one of the coordinate planes, small modifications must be made in the argument). Stokes' theorem can then be applied to each piece of surface, then the separate equalities can be added up to get Stokes' theorem for the whole surface (in the addition, line integrals over the cut-lines cancel out, since they occur twice for each cut, in opposite directions). This completes the argument, *manus undulans*, for Stokes' theorem.

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