**18.02 Practice Final** 3hrs.

**Problem 1.** Given the points \( P : (1,1,-1), Q : (1,2,0), R : (-2,2,2) \) find

a) \( PQ \times PR \)

b) a plane \( ax + by + cz = d \) through \( P, Q \) and \( R \)

**Problem 2.** Let \( \mathbf{A} = \begin{pmatrix} 1 & 0 & c \\ 2 & c & 1 \\ 1 & -1 & 2 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{A}^{-1} = \begin{pmatrix} \vdots \\ \vdots \\ \times \end{pmatrix}. \)

a) For what valu(s) of constant \( c \) will \( \mathbf{A} \mathbf{x} = \mathbf{0} \) have a non-zero solution?

b) Take \( c = 2 \), and tell what entry the inverse matrix has in the position market \( \times \)

**Problem 3.** The roll of Scotch tape has outer radius \( a \) and is fixed in position (i.e., does not turn). Its end \( P \) is originally at the point \( A \); the tape is then pulled from the roll so the free portion makes a 45-degree angle with the horizontal.

Write the parametric equation \( x = x(\theta) \quad y = y(\theta) \) for the curve \( C \) traced out by the point \( P \) as it moves. (Use vectore methods; \( \theta \) is the angle shown)

Sketch the curve on the second picture, showing its behavior at its endpoints.

**Problem 4.** The position vectore of point \( P \) is \( r = < 3 \cos t, 5 \sin t, 4 \cos t >. \)

a) Show its speed is constant.

b) At what point \( A : (a, b, c) \) does \( P \) pass through the \( yz \)-plane?

**Problem 5.** Let \( \omega = x^2y - xy^3 \), and \( P = (2,1) \)

a) Find the directional derivative \( \frac{\partial \omega}{\partial z} \) at \( P \) in the direction of \( \mathbf{A} = 3i + 4j. \)

b) If you start at \( P \) and go a distance .01 in the direction of \( \mathbf{A} \), by approximately how much will \( \omega \) change? (Give a decimal with one significant digit.)

**Problem 6.** a) Find the tangent plane at \( (1,1,1) \) to the surface \( z^2 + 2y^2 + 2z^2 = 5 \); give the equation in the form \( az + by + cz = d \) and simplify the coefficients.

b) What dihedral angle does the tangent plane make with the \( xy \)-plane? (Hint: consider the normal vectors of the two planes.)

**Problem 7.** Find the point on the plane \( 2z + y - z = 6 \) which is closest to the origin, by using Lagrange multipliers. (Minimize the square of the distance. Only 10 points if you use some other method)

**Problem 8.** Let \( \omega = f(x,y,z) \) with the constraint \( g(x,y,z) = 3. \)

At the point \( P : (0,0,0) \), we have \( \nabla f = < 1,1,2 > \) and \( \nabla g = < 2,-1,-1 >. \) Find the value at \( P \) of the two quantities (show work):

a) \( \left( \frac{\partial \omega}{\partial x} \right)_y \)

b) \( \left( \frac{\partial \omega}{\partial y} \right)_y \)
Problem 9. Evaluate by changing the order of integration: \( \int_0^3 \int_0^9 xe^{-y^2} \, dy \, dz \).

Problem 10. A plane region \( R \) is bounded by four semicircles of radius 1. having ends at \((1,1),(1,=1),(−1,1),(−1,−1)\) and centerpoints at \((2,0),(−2,0),(0,2),(0,−2)\).

Set up an iterated integral in polar coordinates for the moment of inertia of \( R \) about the origin; take the density \( δ = 1 \). Supply integrand and limits, but do not evaluate the integral.

Use symmetry to simplify the limits of integration.

Problem 11. a) In the \( xy \)-plane, let \( F = P\mathbf{i} + Q\mathbf{j} \). Give in terms of \( P \) and \( Q \) the line integral representing the flux \( F \) across a simple closed curve \( C \), with outward-pointing normal.

b) Let \( F = axi + byj \). How should the constants \( a \) and \( b \) be related if the flux of \( F \) over any simple closed curve \( C \) is equal to the area inside \( C \)?

Problem 12. A solid hemisphere of radius 1 has its lower flat base on the \( xy \)-plane and center at the origin. Its density function is \( δ = z \). Find the force of gravitational attraction it exerts on a unit mass at the origin.

Problem 13. Evaluate \( \int_C (y-x)dz + (y-z)dz \) over the line segment \( C \) from \( P : (1,1,1) \) to \( Q : (2,4,8) \).

Problem 14. a) Let \( F = ay^2i + 2y(x+z)j + (by^2 + z^2)k \). For what values of the constants \( a \) and \( b \) will \( F \) be conservative? Show work.

b) Using these values, find a function \( f(x,y,z) \) such that \( F = \nabla f \).

c) Using these values, give the equation of a surface \( S \) having the property: \( \int_P^Q \mathbf{F} \cdot dr = 0 \) for any two points \( P \) and \( Q \) on the surface \( S \).

Problem 15. Let \( S \) be the closed surface whose bottom face \( B \) is the unit disc in the \( xy \)-plane and whose upper surface is the paraboloid \( z = 1 - x^2 - y^2 \), \( z \geq 0 \). Find the flux of \( F = xi + yj + zk \) across \( U \) by using the divergence theorem.

Problem 16. Using the data of the preceding problem, calculate the flux of \( F \) across \( U \) directly, by setting up the surface integral for the flux and evaluating the resulting double integral in the \( xy \)-plane.

Problem 17. An \( xz \)-cylinder in 3-space is a surface given by an equation \( f(x,z) = 0 \) in \( x \) and \( z \) alone; its section by any plane \( y = c \) perpendicular to the \( y \)-axis is always the same \( xz \)-curve. Show that if \( F = z^2i + y^2j + xzk \) then \( \int_C \mathbf{F} \cdot dr = 0 \) for any simple closed curve \( C \) lying on an \( xz \)-cylinder. (Use Stokes' theorem)

Problem 18. \( \int e^{-x^2} \, dx \) is not elementary but \( I = \int_0^\infty e^{-x^2} \, dx \) can still be evaluated.

a) Evaluate the iterated integral \( \int_0^\infty \int_0^\infty e^{-x^2} e^{-y^2} \, dy \, dx \), in terms of \( I \).

b) Then evaluate the integral in (a) by switching to polar coordinates. Comparing the two evaluations, what value do you get for \( I \)?