Solutions of Spring 2008 Final Exam

1. (a) The isocline for slope 0 is the pair of straight lines $y = \pm x$. The direction field along these lines is flat.

   The isocline for slope 2 is the hyperbola on the left and right of the straight lines. The direction field along this hyperbola has slope 2.

   The isocline for slope -2 is the hyperbola above and below the straight lines. The direction field along this hyperbola has slope -2.

   (b) The sketch should have the following features:

   The curve passes through $(-2, 0)$. The slope at $(-2, 0)$ is $(-2)^2 - (0)^2 = 4$.

   Going backward from $(-2, 0)$, the curve goes down ($dy/dx > 0$), crosses the left branch of the hyperbola $x^2 - y^2 = 2$ with slope 2, and gets closer and closer to the line $y = x$ but never touches it.

   Going forward from $(-2, 0)$, the curve first goes up, crosses the left branch of the hyperbola $x^2 - y^2 = 2$ with slope 2, and becomes flat when it intersects with $y = -x$. Then the curve goes down and stays between $y = -x$ and the upper branch of the hyperbola $x^2 - y^2 = -2$, until it becomes flat as it crosses $y = x$. Finally, the curve goes up again and stays between $y = x$ and the right branch of the hyperbola $x^2 - y^2 = 2$ until it leaves the box.

   (c) $f(100) \approx 100$

   (d) It follows from the picture in (b) that $f(x)$ reaches a local maximum on the line $y = -x$. Therefore $f(a) = -a$.

   (e) Since we know $f(-2) = 0$, to estimate $f(-1)$ with two steps, the step size is 0.5. At each step, we calculate

   $$x_n = x_{n-1} + 0.5, \qquad y_n = y_{n-1} + 0.5(x_{n-1}^2 - y_{n-1}^2)$$

   The calculation is displayed in the following table.

n	x_n	y_n	$0.5(x_n^2 - y_n^2)$
0	-2	0	2
1	-1.5	2	-0.875
2	-1	1.125	

   The estimate of $f(-1)$ is $y_2 = 1.125$.

2. (a) The equation is $\dot{x} = x(x-1)(x-2)$. The phase line has three equilibria $x = 0, 1, 2$.

   For $x < 0$, the arrow points down.

   For $0 < x < 1$, the arrow points up.

   For $1 < x < 2$, the arrow points down.

   For $x > 2$, the arrow points up.

   (b) The horizontal axis is t and the vertical axis is x. There are three constant solutions $x(t) \equiv 0, 1, 2$. Their graphs are horizontal.

   Below $x = 0$, all solutions are decreasing and they tend to $-\infty$.

Between $x = 0$ and $x = 1$, all solutions are increasing and they approach $x = 1$.
Between $x = 1$ and $x = 2$, all solutions are decreasing and they approach $x = 1$.
Above $x = 2$, all solutions are increasing and they tend to $+\infty$.

(c) A point of inflection $(a, x(a))$ is where $\ddot{x}$ changes sign. In particular, $\ddot{x}(a)$ must be zero. Differentiating the given equation with respect to t, we have

$$\ddot{x} = 2\dot{x} - 6x\dot{x} + 3x^2\dot{x} = \dot{x}(2 - 6x + 3x^2)$$

If $x(t)$ is not a constant solution, $\dot{x}(a) \neq 0$ so that $x(a)$ must satisfy

$$2 - 6x(a) + 3x(a)^2 = 0 \qquad \Leftrightarrow \qquad x(a) = 1 \pm \frac{1}{\sqrt{3}}.$$

(d) <u>Typo in the original version:</u> The material being added into the reactor should be Bo instead of Ct.
Let $x(t)$ be the number of moles of Bo in the reactor at time t. The rate of loading is 2 moles per year. Hence $x(t)$ satisfies $\dot{x} = -kx + 2$, where k is the decay rate of Bo. Since the half life of Bo is 2 years, $e^{-k\cdot 2} = 1/2$ so that $k = (\ln 2)/2$. Therefore we have

$$\dot{x} = -\frac{\ln 2}{2}\, x + 2.$$

The initial condition is $x(0) = 0$.

(e) The differential equation is linear. Since we have

$$y' + \left(\frac{3}{x}\right) y = x$$

an integrating factor is given by

$$\exp\left(\int \frac{3}{x}\, dx\right) = \exp(3\ln x) = x^3.$$

Multiply the above equation by x^3 and integrate:

$$(x^3 y)' = x^3 y' + 3x^2 y = x^4 \qquad \Rightarrow \qquad x^3 y = \frac{1}{5}\, x^5 + c$$

Since $y(1) = 1$, we have $c = 4/5$ and

$$y = \frac{1}{5}\, x^2 + \frac{4}{5}\, x^{-3}.$$

3. (a) Express all complex numbers in polar form:

$$\frac{ie^{2it}}{1+i} = \frac{e^{i\pi/2} e^{2it}}{\sqrt{2} e^{i\pi/4}} = \frac{1}{\sqrt{2}}\, e^{i(2t+\pi/2-\pi/4)} = \frac{1}{\sqrt{2}}\, e^{i(2t+\pi/4)}$$

The real part is

$$\text{Re}\left(\frac{ie^{2it}}{1+i}\right) = \frac{1}{\sqrt{2}}\, \cos\left(2t + \frac{\pi}{4}\right).$$

(b) The trajectory is an outgoing, clockwise spiral that passes through 1.

(c) The polar form of $8i$ is $8e^{i\pi/2}$. Its three cubic roots are

$$\begin{aligned}
2e^{i\pi/6} &= 2\cos\frac{\pi}{6} + 2i\sin\frac{\pi}{6} = \sqrt{3} + i,\\
2e^{i(\pi/6+2\pi/3)} &= 2\cos\frac{5\pi}{6} + 2i\sin\frac{5\pi}{6} = -\sqrt{3} + i,\\
2e^{i(\pi/6+4\pi/3)} &= 2e^{3i\pi/2} = -2i.
\end{aligned}$$

4. (a) Let $x_p(t) = at^2 + bt + c$. Plug it into the left hand side of the equation

$$\begin{aligned}
\ddot{x} + 2\dot{x} + 2x &= (2a) + 2(2at + b) + 2(at^2 + bt + c)\\
&= 2at^2 + (4a + 2b)t + (2a + 2b + 2c)
\end{aligned}$$

and compare coefficients

$$2a = 1, \qquad 4a + 2b = 0, \qquad 2a + 2b + 2c = 1.$$

The solution is $a = 1/2$, $b = -1$, $c = 1$. Therefore $x_p(t) = \frac{1}{2}t^2 - t + 1$.

(b) The characteristic polynomial is $p(s) = s^2 + 2s + 2$. Using the ERF and linearity,

$$x_p(t) = \frac{e^{-2t}}{p(-2)} + \frac{1}{p(0)} = \frac{e^{-2t}}{2} + \frac{1}{2}$$

(c) Consider the complex equation

$$\ddot{z} + 2\dot{z} + 2z = e^{it}.$$

For any solution z_p, its imaginary part $x_p = \operatorname{Im} z_p$ satisfies the real equation

$$\ddot{x} + 2\dot{x} + 2x = \sin t.$$

The ERF provides a particular solution of the complex equation

$$z_p(t) = \frac{e^{it}}{p(i)} = \frac{e^{it}}{1+2i} = \frac{e^{it}}{\sqrt{5}\,e^{i\phi}} = \frac{1}{\sqrt{5}}\,e^{i(t-\phi)}$$

where ϕ is the polar angle of $1 + 2i$. Take the imaginary part of z_p

$$x_p(t) = \operatorname{Im} z_p(t) = \frac{1}{\sqrt{5}}\sin(t - \phi)$$

This is a sinusoidal solution of the real equation. Its amplitude is $1/\sqrt{5}$.

(d) If $x(t) = t^3$ is a solution, then $q(t) = \ddot{x} + 2\dot{x} + 2x = 6t + 6t^2 + t^3$.

(e) The general solution is $x(t) = t^3 + x_h(t)$, where $x_h(t)$ is a solution of the associated homogeneous equation. Since the characteristic polynomial $s^2 + 2s + 2$ has roots $-1 \pm i$,

$$x(t) = t^3 + x_h(t) = t^3 + c_1e^{-t}\cos t + c_2e^{-t}\sin t.$$

5. (a) See the formula sheet for the definition of sq(t). The graph of $f(t)$ is a square wave of period 2π. It has a horizontal line segment of height 1 in the range $-\pi/2 < t < \pi/2$ and a horizontal line segment of height -1 in the range $\pi/2 < t < 3\pi/2$.

(b) Replace t by $t + \pi/2$ in the definition of sq(t)

$$\begin{aligned} f(t) = \mathrm{sq}\left(t + \frac{\pi}{2}\right) &= \frac{4}{\pi}\left[\sin\left(t + \frac{\pi}{2}\right) + \frac{1}{3}\sin\left(3t + \frac{3\pi}{2}\right) + \frac{1}{5}\sin\left(5t + \frac{5\pi}{2}\right) + \dots\right] \\ &= \frac{4}{\pi}\left(\cos t - \frac{1}{3}\cos 3t + \frac{1}{5}\cos 5t + \dots\right) \end{aligned}$$

(c) First consider the complex equation

$$\ddot{z} + z = e^{int} \qquad \text{for a positive integer } n.$$

The characteristic polynomial is $p(s) = s^2 + 1$. One of the ERFs provides a particular solution of the complex equation

$$\begin{aligned} z_p(t) &= \frac{e^{int}}{p(in)} = \frac{e^{int}}{1 - n^2}\,, \qquad n \neq 1 \\ z_p(t) &= \frac{te^{it}}{p'(i)} = \frac{te^{int}}{2i}\,, \qquad n = 1 \end{aligned}$$

The imaginary parts of these functions

$$\begin{aligned} u_p(t) &= \mathrm{Im}\left(\frac{e^{int}}{1 - n^2}\right) = \frac{\sin nt}{1 - n^2}\,, \qquad n \neq 1 \\ u_p(t) &= \mathrm{Im}\left(\frac{te^{it}}{2i}\right) = -\frac{1}{2}t\cos t\,, \qquad n = 1 \end{aligned}$$

satisfy the imaginary part of the above complex equation, namely

$$\ddot{u} + u = \sin nt.$$

By linearity, a solution of $\ddot{x} + x = \mathrm{sq}(t)$ is given by

$$x_p(t) = \frac{4}{\pi}\left(-\frac{1}{2}t\cos t + \frac{1}{3}\cdot\frac{\sin 3t}{1 - 3^2} + \frac{1}{5}\cdot\frac{\sin 5t}{1 - 5^2} + \dots\right).$$

6. (a) For $t < 0$, the graph is flat on t-axis.
For $0 < t < 1$, the graph is flat at 1 unit above t-axis.
For $1 < t < 3$, the graph is flat at 1 unit below t-axis.
For $3 < t < 4$, the graph is flat at 1 unit above t-axis.
For $t > 4$, the graph is flat on t-axis.

(b)
$$\begin{aligned} v(t) &= [u(t) - u(t-1)] - [u(t-1) - u(t-3)] + [u(t-3) - u(t-4)] \\ &= u(t) - 2u(t-1) + 2u(t-3) - u(t-4) \end{aligned}$$

(c) The graph coincides with t-axis for all t, except for two upward spikes at $t = 0, 3$ and two downward spikes at $t = 1, 4$.

(d) $\dot{v}(t) = \delta(t) - 2\delta(t-1) + 2\delta(t-3) - \delta(t-4)$

(e) By the fundamental solution theorem (a.k.a. Green's formula),

$$x(t) = (q * w)(t) = \int_0^t q(t-\tau)w(\tau)\,d\tau = \int_{a(t)}^{b(t)} w(\tau)\,d\tau.$$

Now $q(t-\tau) = 1$ only for $0 < t-\tau < 1$, or $t-1 < \tau < t$, and it is zero elsewhere. Therefore the upper limit $b(t)$ equals t. The lower limit $a(t)$ is $t-1$ if $t-1 > 0$, or 0 if $t-1 < 0$. In other words, $a(t) = (t-1)u(t-1)$.

7. (a) The transfer function is $W(s) = \dfrac{1}{p(s)} = \dfrac{1}{2s^2 + 8s + 16}$.

(b) The unit impulse response $w(t)$ is the inverse Laplace transform of $W(s)$. In other words,

$$\begin{aligned} \mathcal{L}(w(t)) &= \frac{1}{2s^2+8s+16} = \frac{1}{2[(s+2)^2+4]} \\ \Rightarrow \quad \mathcal{L}(e^{2t}w(t)) &= \frac{1}{2(s^2+4)} = \frac{1}{4}\mathcal{L}(\sin 2t) \end{aligned}$$

Therefore $e^{2t}w(t) = \dfrac{1}{4}\sin 2t$, and $w(t) = \dfrac{1}{4}e^{-2t}\sin 2t$.

(c) Take the Laplace transform of

$$p(D)x = 2\ddot{x}(t) + 8\dot{x}(t) + 16x(t) = \sin t$$

with the initial conditions $x(0+) = 1$, $\dot{x}(0+) = 2$. This yields

$$\begin{aligned} &2[s^2X(s) - s - 2] + 8[sX(s) - 1] + 16X(s) = \frac{1}{s^2+1} \\ \Rightarrow \quad &X(s) = \frac{1}{2s^2+8s+16}\left(\frac{1}{s^2+1} + 2s + 12\right) \end{aligned}$$

8. (a) The characteristic polynomial of A is

$$\det(A - \lambda I) = \det\begin{bmatrix} 2-\lambda & 12 \\ 3 & 2-\lambda \end{bmatrix} = (2-\lambda)^2 - 36 = (\lambda - 8)(\lambda + 4).$$

Therefore the eigenvalues are $\lambda = 8, -4$.

(b) For $\lambda = 8$, solve $(A - 8I)\mathbf{v} = \mathbf{0}$. Since $A - 8I = \begin{bmatrix} -6 & 12 \\ 3 & -6 \end{bmatrix}$, a solution is $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

For $\lambda = -4$, solve $(A + 4I)\mathbf{v} = \mathbf{0}$. Since $A + 4I = \begin{bmatrix} 6 & 12 \\ 3 & 6 \end{bmatrix}$, a solution is $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

(c) The following is a fundamental matrix for $\dot{\mathbf{u}} = B\mathbf{u}$

$$F(t) = \begin{bmatrix} e^t & -e^{2t} \\ e^t & e^{2t} \end{bmatrix}$$

Then e^{tB} can be computed as $F(t)F(0)^{-1}$.

$$F(0) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \qquad F(0)^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$e^{tB} = F(t)F(0)^{-1} = \begin{bmatrix} e^t & -e^{2t} \\ e^t & e^{2t} \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^t + e^{2t} & e^t - e^{2t} \\ e^t - e^{2t} & e^t + e^{2t} \end{bmatrix}$$

(d) The general solution of $\dot{\mathbf{u}} = B\mathbf{u}$ is

$$\mathbf{u}(t) = c_1 \begin{bmatrix} e^t \\ e^t \end{bmatrix} + c_2 \begin{bmatrix} -e^{2t} \\ e^{2t} \end{bmatrix} = F(t) \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

The given initial condition implies

$$\begin{aligned} \begin{bmatrix} 2 \\ 1 \end{bmatrix} &= F(0)\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ \Rightarrow \quad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} &= F(0)^{-1}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix} \end{aligned}$$

Therefore the solution of the initial value problem is $\mathbf{u}(t) = \frac{1}{2}\begin{bmatrix} 3e^t + e^{2t} \\ 3e^t - e^{2t} \end{bmatrix}$.

9. (a) The phase portrait has the following features:
   - All trajectories start at $(0, 0)$ and run off to infinity.
   - There are straight line trajectories along the lines $y = \pm x$.
   - All other trajectories are tangent to $y = x$ at $(0, 0)$.
   - No two trajectories cross each other.

   (b) $\operatorname{Tr} A = a + 1, \quad \det A = a + 4, \quad \Delta = (\operatorname{Tr} A)^2 - 4(\det A) = (a - 5)(a + 3)$
   - (i) $\det A < 0 \quad \Leftrightarrow \quad a < -4$
   - (ii) not for any a
   - (iii) $\Delta > 0$, $\operatorname{Tr} A < 0$ and $\det A > 0 \quad \Leftrightarrow \quad -4 < a < -3$
   - (iv) $\Delta < 0$ and $\operatorname{Tr} A < 0 \quad \Leftrightarrow \quad -3 < a < -1$; counterclockwise
   - (v) $\Delta < 0$ and $\operatorname{Tr} A > 0 \quad \Leftrightarrow \quad -1 < a < 5$
   - (vi) $\Delta = 0$ and $\operatorname{Tr} A > 0 \quad \Leftrightarrow \quad a = 5$

10. (a) The equilibria are the solutions of

$$\dot{x} = x^2 - y^2 = 0, \qquad \dot{y} = x^2 + y^2 - 8 = 0.$$

This implies $(x^2, y^2) = (4, 4)$, so that $(x, y) = (2, 2), (2, -2), (-2, 2), (-2, -2)$.

(b) The Jacobian is $J(x, y) = \begin{bmatrix} 2x & -2y \\ 2x & 2y \end{bmatrix}$. In particular, $J(-2, -2) = \begin{bmatrix} -4 & 4 \\ -4 & -4 \end{bmatrix}$.

(c) The linearization of the nonlinear system at $(-2,-2)$ is the linear system $\dot{\mathbf{u}} = J(-2,-2)\mathbf{u}$. A computation shows that the eigenvalues of $J(-2,-2)$ are $-4 \pm 4i$. The first component of $\mathbf{u}(t)$ is of the form

$$c_1 e^{-4t} \cos 4t + c_2 e^{-4t} \sin 4t = A e^{-4t} \cos(4t - \phi).$$

This means $x(t) \approx -2 + Ae^{-4t} \cos(4t - \phi)$ near $(-2,-2)$.

(d) Let $f(x) = 2x - 3x^2 + x^3$. The phase line in problem 2(a) shows that $\dot{x} = f(x)$ has a stable equilibrium at $x = 1$.

The linearization of the nonlinear equation at $x = 1$ is the linear equation $\dot{u} = f'(1)u = -u$. Its solutions are $u(t) = Ae^{-t}$. This means $x(t) \approx 1 + Ae^{-t}$ near $x = 1$.