18.03 Lecture 26, April 9, 2010

Laplace Transform: definition and basic properties

1. Definition of LT; L[1]
2. Region of convergence
3. Powers
4. Linearity
5. s-shift rule
6. sines and cosines
7. t-domain and s-domain

[1] Laplace Transform

We continue to consider functions f(t) which are zero for t < 0 .
(I may forget to multiply by u(t) now and then.)

The Laplace transform takes a function f(t) (of "time") and uses it to manufacture another function F(s) (where s can be complex). It: [Slide]

(1) makes explicit long term behavior of f(t) .

(2) answers the question: if I know w(t), how can I compute p(s) ?

(3) converts differential equations into algebraic equations.

But we won't see these virtues right away.

Definition: The Laplace transform of f(t) is the improper integral

F(s) = integral_{0}^¥infty e^{-st} f(t) dt

(formula subject to two refinements).

We will often write f(t) ------> F(s)

and L[f(t)] = F(s)

(This notation isn't so good, because there's no room for "s" on the left.)

For each value of s , F(s) is a weighted integral of f(t).
Wnen s = 0 , for example, the Laplace integral is just the integral of f(t). When s > 0 , the values of f(t) for large t are given less weight. Each value of F(s) contains information about the whole of f(t).

Example: f(t) = 1 :

F(s) = integral_0^infty e^{-st} dt

= lim_{T --> infty} e^{-st}/(-s) |^T_0

= (-1/s) (lim_{T --> infty} e^{-st} - 1).

If s > 0 , e^{-st} ---> 0 as T ---> infty , and we get 1/s .

If s < 0 , e^{-st} ---> infty as T ---> infty , and the improper integral diverges.

Actually, I'll want s to be a complex number, not just a real number. With s = a+bi , !e^{st}| = |e^{at}e^{ibt}| = e^{at} , and the same argument shows that the integral converges for Re(s) > 0 .

So this is our first calculation:

1 has Laplace transform 1/s

[2] For a more general function f(t) , the integral will converge for a given value of s provided that |f(t)e^{-st}| can be integrated. This will happen if

|f(t)| < |e^{st}| = e^{(Re s)t}

at least for t large. So to make the integral converge *somewhere*, we make the assumption that

f(t) is "of exponential order" if there is a constant M such that for large t , |f(t)| < e^{at} .

In that case, the integral for F(s) converges for Re(s) >= a .

Eg e^{t^2} has no Laplace transform.

So in the definition I really meant:

Refinement #1: "for Re(s) large." -- the "Region of Convergence."

We'll see that the region of convergence contains important information and it's good practice to declare it when you state a Laplace transform:

1 has LT 1/s with region of convergence Re(s) > 0 .

The expression obtained by means of the integration makes sense everywhere in C except for a few points -- like s = 0 here -- and this is how we define the Laplace transform for values of s whose real part is too small. This is called "analytic continuation."

[3] Powers

L[t^n] = int_0^infty t^n e^{-st} dt

u = t^n dv = e^{-st} dt
du = n t^{n-1} dt v = (-1/s) e^{-st}

= t^n (-1/s) e^{-st} |_0^infty + (n/s) int_0^infty t^{n-1} e^{-st} dt

= n/s L[t^{n-1}] .

Start with n = 1 : L[t] = (1/s)(1/s) = 1/s^2 .

n = 2 : L[t^2] = (2/s)(1/s^2) = 2/s^3

$n = 3$: $L[t^3] = (3/s)(2/s^3) = 3!/s^4$

... $L[t^n] = n!/s^{n+1}$.

The region of convergence is $Re(s) > 0$.

[4] These first computations can be exploited using general properties of the
Laplace Transform. We'll develop quite a few of these rules, and in fact normally you will not be using the integral definition to compute Laplace transforms.

Rule 1 (Linearity): $af(t) + bg(t) \longrightarrow aF(s) + bG(s)$.

This is clear from the linearity of the integral.

Question 26.1. What is $L[(1+t)^2]$?

1. $(1/s + 1/s^2)^2 = 1/s^2 + 2/s^3 + 1/s^4$
2. $1/s + 2/s^2 + 2/s^3$
3. $(1+t)(1/s + 1/s^2)$
4. Don't know

[5] Rule 2 (s-shift): If z is any complex number,

$$L[e^{zt}f(t)] = F(s-z) \text{ with region of convergence } Re(s) > Re(z)$$

Here's the calculation:

$$L[e^{zt}f(t)] = \int_0^\infty e^{zt} f(t) e^{-st}\, dt$$

$$= \int_0^\infty f(t) e^{-(s-z)t}\, dt$$

$$= F(s-z).$$

Using $f(t) = 1$ and our calculation of its Laplace transform we find

$$L[e^{zt}] = 1/(s-z). \qquad (*)$$

[6] As usual we can get sinusoids out from the complex exponential.

Using linearity and Euler's identity

$$\cos(\omega t) = (1/2)(e^{i \omega t} + e^{-i \omega t})$$

we find

$$L[\cos(\omega t)] = (1/2)((1/(s - i \omega) + 1/(s + i \omega))$$

$$= s/(s^2 + \omega^2)$$

Using

    sin(omega t) = (1/2i)(e^{i omega} - e^{-i omega})

we find

    L[sin(omega t)] = omega/(s^2 + omega^2).

Both are convergent for Re(s) > 0 , since +- i omega t lie on the imaginary axis.

[7] Two worlds ... [Slide]

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|                                                             |
| The t domain                                                  |
|                                                             |
| t  is real and positive                                         |
|                                                             |
| functions  f(t)  are signals, i.e. functions, perhaps generalized     |
|                                                             |
| ODEs relating them                                             |
|                                                             |
| convolution                                                   |
|                                                             |

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                            |      ^
                         L  |      | L^{-1}
                            v      |

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|                                                             |
| The s domain                                                  |
|                                                             |
| s  is complex                                                 |
|                                                             |
| beautiful functions  F(s) ,  often rational = poly/poly             |
|                                                             |
| and algebraic equations relating them                               |
|                                                             |
| ordinary multiplication of functions                                |
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