The matrix exponential: initial value problems.

1. Definition of $e^{At}$
2. Computation of $e^{At}$
3. Uncoupled example
4. Defective example
5. Exponential law

[1] Recall from day one:

(a) $x' = rx$ with initial condition $x(0) = 1$ has solution $x = e^{rt}$.

Later, we decided to *define* $e^{it}$ as the solution to

(b) $x' = ix$ with initial condition $x(0) = 1$.

Following Euler, a solution is given by $\cos t + i \sin t$, so we found that $e^{it} = \cos(t) + i \sin(t)$.

(c) Now we are studying $u' = Au$. Let's try to *define*

The solution to $u' = Au$ with initial condition $u(0)$ is $u = e^{At}u(0)$. (**)

Note that the initial value $u(0)$ is a vector, and $u(t)$ is a vector valued function. So the expression $e^{At}$ must denote a matrix, or rather a matrix valued function.

What could $e^{At}$ be? For a start, what is its first column?

Recall that the first column of any matrix $B$ is the product $B[1;0]$, and $\begin{bmatrix} b_1 & b_2 \end{bmatrix}[1;0] = b_1$, so combining this with (**) we see:

The first column of $e^{At}$ is the solution to $u' = Au$ with $u(0) = [1;0]$.

Similarly,

The second column of $e^{At}$ is the solution to $u' = Au$ with $u(0) = [0;1]$.

This is the DEFINITION of $e^{At}$. It makes (**) true for all $u(0)$, because $e^{At}u(0)$ is a solution (being a linear combination of the columns of $e^{At}$, which are solutions), and when $t = 0$ we get $e^{A0}u(0) = Iu(0) = u(0)$.

[2] Computation of $e^{At}$

We need a method for computing it, though. To explore this we'll use the
Example: \( A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \).

This is upper triangular, so its eigenvalues are given by the diagonal entries: \( \lambda_1 = 1 \), \( \lambda_2 = 2 \). The \((\text{tr}, \text{det})\) pair lies in the upper right quadrant, below the critical parabola; the phase portrait is an unstable node.

Find eigenvectors:

\[
\begin{align*}
\lambda_1 &= 1 : \quad A - I : \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix}?;?\end{bmatrix} = \begin{bmatrix}0;0\end{bmatrix} : \quad v_1 = \begin{bmatrix}1;0\end{bmatrix} \\
\lambda_2 &= 2 : \quad A - 2I : \begin{bmatrix}-1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix}?;?\end{bmatrix} = \begin{bmatrix}0;0\end{bmatrix} : \quad v_2 = \begin{bmatrix}1;1\end{bmatrix}
\end{align*}
\]

Two independent solutions are given by

\[
\begin{align*}
u_1 &= \begin{bmatrix} e^t \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix}
\end{align*}
\]

and the general solution is

\[
u = c_1 \begin{bmatrix} e^t \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix}
\]

We could go ahead and solve for \( c_1 \) and \( c_2 \) to get solutions with the desired initial conditions. What follows is a clever way to do that.

There is a compact way to write this linear combination: it is

\[
u = \begin{bmatrix} e^t , e^{2t} \\ 0 , e^{2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}. \quad (***)
\]

This matrix is a "fundamental matrix" for the system: its columns are independent solutions. Such a matrix will be denoted by \( \Phi(t) \); so here

\[
\Phi(t) = \begin{bmatrix} e^t , e^{2t} \\ 0 , e^{2t} \end{bmatrix}
\]

\( \Phi(t) \) behaves very much like we want \( e^{At} \) to behave; its columns are solutions, even independent ones, and the general solution is given by

\[
\Phi(t) \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}
\]

The matrix exponential \( e^{At} \) is a fundamental matrix: it is the fundamental matrix \( \Phi(t) \) such that \( \Phi(0) = I \).

Our \( \Phi(t) \) does not evaluate this way. To fix this, I claim we should form

\[
\Phi(t) \Phi(t)^{-1}
\]

Explanation: If \( B \) is a square matrix, you can ask whether it has an *inverse* matrix, a matrix \( B^{-1} \) such that

\[
B B^{-1} = I \quad \text{and} \quad B^{-1} B = I
\]

(either implies both). The answer, as for numbers, is not always. It turns out that there is an inverse exactly when \( \det(B) \) is not zero.

In the 2x2 case, \( B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) and

\[
\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.
\]
We can check this:

\[
\begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix}
\begin{bmatrix}
  d & -b \\
  -c & a
\end{bmatrix}
= \begin{bmatrix}
  ad-bc & 0 \\
  0 & ad-bc
\end{bmatrix}
= (\det B) I
\]

Now let's see: each column \( \Phi(t) \Phi(0)^{-1} \) is a linear combination of the columns of \( \Phi(t) \), so it's a solution. What remains is to check the normalization; but \( \Phi(0) \Phi(0)^{-1} = I \).

Conclusion:

\[
e^{A t} = \Phi(t) \Phi(0)^{-1}
\]

where \( \Phi(t) \) is ANY fundamental matrix for \( A \).

In our example, \( \Phi(0) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \), \( \Phi(0)^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \), and so

\[
e^{A t} = \begin{bmatrix} e^t & e^{2t} \\ 0 & e^{2t} \end{bmatrix}
\begin{bmatrix}
  1 & -1 \\
  0 & 1
\end{bmatrix}
= \begin{bmatrix}
  e^t , e^{2t} - e^t \\
  0 , e^{2t}
\end{bmatrix}.
\]

[3] Uncoupled example: Suppose \( A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \).
The eigenvalues are \( \lambda_1 = a \) and \( \lambda_2 = d \). I can see the eigenvectors: \( v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \), \( v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \).

Basic solutions are \( e^{\lambda_1 t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \)
\( e^{\lambda_2 t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \)
so \( \Phi(t) = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \)

and this is already normalized: so it is the matrix exponential.


Sometimes the matrix exponential can be a bit unexpected. For example:

\[
A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}
\]

Then \( \text{tr} A = 0 \) and \( \det A = 0 \), so the only eigenvalue is 0, with multiplicity 2. This is not a diagonal matrix, so it is defective, and we could find solutions by the standard method. However, it is also a companion matrix, for the second order equation \( x'' = 0 \). Solutions of this are easy! \( x_1 = 1 \), \( x_2 = t \). So basic solutions to

\[
u' = A u
\]
are \( u_1 = \begin{bmatrix} x_1 \\ x_1' \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \)
\( u_2 = \begin{bmatrix} x_2 \\ x_2' \end{bmatrix} = \begin{bmatrix} t \\ 1 \end{bmatrix} \)

\[
\Phi(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}
\]
This satisfies \( \Phi(0) = I \), so
\[ e^{\{At\}} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \]

[5] We can take \( t \) to be a specific value, of course: eg \( t = 1 \):

\[ e^{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \]

and this lets us define \( e^A \) for any square matrix \( A \).

Then \( e^0 = I \), as you might expect, but watch out:

\[ e^A e^B = e^{\{A+B\}} \quad \text{*provided that*} \quad AB = BA \]

So for example \( (e^A)^n = e^{\{nA\}} \)