Section II Solutions

2A-1) This is true because \( D^2, D, \) and \( \frac{d}{dx} \) are all linear operators.
\[
q(y_1+y_2) = qy_1 + qy_2 \quad (a) \\
pD(y_1+y_2) = p(Dy_1 + Dy_2) \quad (b) \\
pDy_1 + pDy_2 \quad (c) \\
D^2(y_1+y_2) = D^2y_1 + D^2y_2 \quad (d)
\]

Adding (b) & (c) gives
\[
L(y_1 + y_2) = Ly_1 + Ly_2 
\]
The proof for \( L(cy_1) = cLy_1 \) is similar.

b) (i) \( L(y_0) = 0 \) since \( y_0 \) solves the eqn \( Ly_0 = 0 \)
\[
\text{Adding } L(y_1 + y_2) = 0 \quad \Rightarrow \quad y_1 + y_2 \text{ is a soln.}
\]

(ii) If \( y_1 \) is any soln, then
\[
L(y_1 - y_2) = Ly_1 - Ly_2 = r - r = 0
\]
\[
y_1 - y_2 = y_0 \quad \text{(a soln of } Ly_0 = 0) \\
y_1 = y_0 + y_2
\]
Parts (i) + (ii) together show all solns are of the form \( y_0 + y_2 \).

2A-2a) \( y = c_1 e^{x} + c_2 e^{2x} \) \( y' = c_1 e^{x} + 2c_2 e^{2x} \) \( y'' = c_2 e^{x} + 4c_2 e^{2x} \)
\[
\begin{align*}
y' - y &= c_2 e^{2x} \\
y'' - 3y + 2y &= 0 
\end{align*}
\]

b) The question is whether we can find values for \( c_1, c_2 \) such that
\[
\begin{align*}
c_1 e^{x} + c_2 e^{2x} &= y_0 \\
\end{align*}
\]

These equations can be solved by Cramer's rule (coefficient determinant)
for \( c_1, c_2 \) provided that
\[
\begin{vmatrix} e^{x} & e^{2x} \\ e^{x} & e^{2x} \end{vmatrix} \neq 0
\]
But this det \( = e^{3x} \) \( \neq 0 \) for any \( x_0 \).

2A-3a) \( y = c_1 x + c_2 x^2 \) \( y' = c_1 + 2c_2 x \) \( y'' = 2c_2 \)

You want \( x \)
\[
\begin{align*}
y' &= c_1 + 2c_2 x \\
y'' &= 2c_2
\end{align*}
\]

One way:
\[
\begin{align*}
c_2 &= y'' \\
c_1 &= y' - 2c_2 x
\end{align*}
\]

From last eqn
\[
y' = y'' - 2c_2 x \text{ from } 2^{nd} \text{ & } 3^{rd} \text{ eqn.}
\]

Substitute into 1^{st} eqn, get
\[
y = (y - y'' - 2c_2 x) x + \frac{y''}{2} x^2
\]

which by algebra becomes
\[
x^2y'' - 2xy' + 2y = 0
\]

b) All solns \( y = c_1 x + c_2 x^2 \)

Satisfy \( y(0) = 0 \)

c) This theorem requires that when eqn is written \( y'' + p(x)y' + q(x)y = 0 \), that \( p, q \) be continuous functions.

But here, the ODE in standard form is
\[
y'' - \frac{3}{2} y' + \frac{3}{4} y = 0
\]
coefficients are discontinuous at \( x = 0 \).

2A-4a) Suppose \( y_1 \) is a solution to
\[
y'' + p(x)y' + q(x)y = 0 \quad \text{(with the same property)}
\]

Then \( y_1(x_0) = 0 \)
\[
y_1'(x_0) = 0
\]

But \( y_2(x) = 0 \) is another soln to
\[
y'' + p(x)y' + q(x)y = 0
\]

Part (a) is not contradicted, since the coefficient \( \frac{1}{x} \) is discontinuous at \( x = 0 \).
2A-5 a) \( W(e^{mx}, e^{nx}) = \begin{vmatrix} e^{mx} & e^{nx} \\ me^{mx} & me^{nx} \end{vmatrix} = (m_2-m_1)e^{(m_2-m_1)x} \)

Since \( e^x \neq 0 \) for all \( x \), this is never 0.

If \( m_1 \neq m_2 \), \( W \) is linearly independent.

b) \( W(e^{mx}, xe^{mx}) = \begin{vmatrix} e^{mx} & xe^{mx} \\ me^{mx} & me^{mx} + e^{mx} \end{vmatrix} = e^{2mx} \neq 0 \) for any \( x \).

(This holds true even if \( m = 0 \).)

\( \therefore \) The functions are linearly independent.

2A-6 (The graph of \( x|x| = \begin{cases} 5x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases} \)

a) If \( x \geq 0 \), \( W = \frac{x^2 - x^2}{2x} = 0 \)

If \( x \leq 0 \), \( W = \frac{x^2 - x^2}{-2x} = 0 \)

b) Suppose they were linearly dependent on an interval \( (a, b) \) containing 0, that is, suppose there are \( c_1, c_2 \) such that

\[ c_1 y_1 + c_2 y_2 = 0 \quad \text{for all } x \in (a, b) \]

Then if \( x > 0 \), \( y_1 = y_2 \), \( \therefore c_1 = -c_2 \)

if \( x < 0 \), \( y_1 = -y_2 \), \( \therefore c_1 = c_2 \)

Thus \( c_1 = 0 \) and \( c_2 = 0 \), so that \( y_1 \) and \( y_2 \) are not linearly dependent on \( (a, b) \).

Since \( y_2 = 2x \) for \( x > 0 \),

\( y_2 = -2x \) for \( x < 0 \)

graph \( y, y' \) is \( \) \( \)

Thus \( y'' \) does not exist at \( x = 0 \),

so it cannot be the solution to a 2nd order equation \( y'' + p(x)y' + q(x)y = 0 \)
on the interval \( (a, b) \) containing 0.

\( W(x, W) = \) not contradicted.

2A-7 (a) This can be done directly, by differentiating \( y_1, y_2 - y_2, y_1 \) (see below).

An elegant way to do it is to use the formula for differentiating a determinant: differentiate one at a time, then add:

\[ \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} = \begin{vmatrix} u_1' & u_2' \\ v_1' & v_2' \end{vmatrix} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \]

(this works for sets of any size).

Applying this to the Wronskian:

\[ \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} + \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \]

since \( y_1 \) and \( y_2 \) solve \( y'' = -py' - qy \),

we get the above right-hand side:

\[ \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} - \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \]

(a) \( p \) being constant.

(b) From part (a), if \( p(x) = 0 \),

then \( \frac{dW}{dx} = 0 \), so \( W(y_1, y_2) = c \).

c) \( y'' + ky = 0 \)

Here \( p = 0 \)

\[ W(cos kx, sin kx) = \begin{vmatrix} cos kx & sin kx \\ sin kx & cos kx \end{vmatrix} = k(cos^2 kx + sin^2 kx) = k \]

a constant.
28-1

a) \( y_2 = u e^x \)

\[
\begin{align*}
y_2' &= u' e^x + u e^x \\
y_2'' &= u'' e^x + 2u' e^x + u e^x
\end{align*}
\]

Multiply second row by \(-2\) and add:

\[
y_2'' - 2y_2' + y_2 = u'' e^x \quad \text{(all other terms)}
\]

If \( y_2 \) is a soln to the ODE, the left-hand side must be 0. Therefore we must have

\[
u'' e^x = 0
\]

so

\[
u'' = 0,
\]

\[
u = ax + b
\]

and

\[
y_2 = (ax + b)e^x
\]

Any of these which \( \neq 0 \) give a second solution - for \( \frac{d^2y_2}{dx^2} \)

b) From II/7a, \( \frac{dW}{dx} = -P W = 2W \)

\[
\because W(y_1, y_2) = ce^{2x}, \quad c \neq 0
\]

But \( W(y_1, y_2) = \begin{vmatrix} e^x & y_2 \\ e^x & y_2' \end{vmatrix} \)

Equating these two expressions for \( W \),

\[
e^x(y_2' - y_2) = ce^{2x}
\]

\[
\therefore y_2' - y_2 = ce^x
\]

(c can have any \( \neq 0 \) value)

Solving this ODE gives (it's a linear equation)

\[
y_2 = e^x [(x + c_1) \quad \text{as a family of second solutions}.
\]

\[
c) \quad y_2 = e^x \int \frac{1}{e^x} e^{-\int 2dx} \ dx
\]

\[
= e^x \int 1 \ dx = e^x (x + c)
\]

[more generally: \( e^{\int 2dx} = e^{2x+c_2} \),

\[
y_2 = e^x \int (c_2) dx \quad \text{put } c_2 = c_1
\]

\[
= e^x (x + c_1)
\]

A second soln

(28-2)

\[
W(y_1, y_2) = \begin{vmatrix} e^x & e^{(ax+b)} \\ e^x & e^{(ax+b) + ae^x} \end{vmatrix}
\]

\[
= ae^{2x} \neq 0 \quad \text{if } a \neq 0
\]

[This shows it's the special equation only].

In general:

\[
W(y_1, y_2) = y_1 y_2' - y_1 y_2
\]

\[
y_2' = y_1 \int \frac{1}{y_1} e^{\int dx} \ dx
\]

\[
= \frac{1}{y_1} \int e^{\int dx} + y_1 \frac{1}{y_1} e^{-\int dx}
\]

\[
= y_1 \int \frac{\text{dy}}{y_1} + \frac{1}{y_1} e^{-\int dx}
\]

\[
\therefore W(y_1, y_2) = y_1 y_2' + e^{-\int dx} - y_1 y_2
\]

\[
= e^{-\int dx} \neq 0
\]

[Note that this same formula for the Wronskian follows from II/7a].

(28-3)

Let \( y_2 = x > u \), so that

\[
y_2' = u + xu', \quad y_2'' = 2u' + xu''
\]

Substituting into \( x^2 y'' + 2xy' - 2y = 0 \)

gives after cancellation and dividing by \( x^2 \):

\[
x u'' + 4u' = 0
\]

Put \( v = u' \),

\[
x \frac{dv}{dx} + 4v = 0 \quad \text{or } \frac{dv}{v} = -\frac{4}{x} \ dx
\]

Solving,

\[
\frac{v}{x} = \frac{C_1}{x^4}, \quad \text{and } u' = \frac{C_1}{x^4}
\]

\[
u = \frac{C_1}{x^3} + c_0 = \frac{C_1}{x^3} + C_0
\]

\[
y_2 = \left[ \frac{C_1}{x^3} + C_0 x \right], \quad \text{a second soln}
\]

[can also use the general formula given in II/8c]

(28-4)

Using the general formula [II/8c]:

Find: \( e^{\int 2dx} \)

\[
= \int \frac{1}{x^2} e^{\int dx} = \ln(1-x^2)
\]

\[
\therefore \int \frac{1}{x^2} e^{\int dx} = \int \frac{dx}{x^2(1-x^2)}
\]

we do this by partial fraction \( \rightarrow \) (cont'd)
(cont'd)

\[ \frac{1}{x^2(1-x^2)} = \frac{1}{x^3(1-x)(1+x)} = \frac{1}{x^2} + \frac{1}{2} \frac{1}{1-x} + \frac{1}{2} \frac{1}{1+x} \]

\[ \therefore \int \frac{dx}{x^3(1-x^2)} = -\frac{1}{x} + \frac{1}{2} \ln(1-x) + \frac{1}{2} \ln(1+x) \]

\[ y_2 = y_1 \int \frac{1}{y_1^2} e^{-spdx} = -\frac{1}{x} + \frac{1}{2} \ln(1-x) \]

The general solution is now

\[ c_1y_1 + c_2y_2 \]

or

\[ c_1x + c_2 \left( -1 + \ln \frac{1-x}{1+x} \right) \]

a) Char eqn: \( \lambda^2 - 3\lambda + 2 = 0 \)

\( \therefore (\lambda - 1)(\lambda - 2) = 0 \)

roots: \( \lambda = 1, 2 \)

\[ y = c_1e^x + c_2e^{2x} \]

b) Char eqn: \( r^2 + 2r - 3 = 0 \)

\( (r + 3)(r - 1) = 0 \)

\[ y = c_1e^x + c_2e^{-3x} \]

Put in initial conditions:

\[ y(0) = 1 \Rightarrow c_1 + c_2 = 1 \]

\[ y'(0) = 1 \Rightarrow c_1 - 3c_2 = -1 \]

\( c_1 = \frac{1}{2}, \quad c_2 = \frac{1}{2} \)

\[ y = \frac{1}{2} e^x + \frac{1}{2} e^{-3x} \]

c) Char eqn: \( r^2 + 2r + 2 = 0 \)

By quad. formula: \( r = -1 \pm i \)

\[ y = e^{-x} (c_1 \cos x + c_2 \sin x) \]

[using as \( y_1, y_2 \) the real + imaginary parts of the exp. soln]

\[ y = e^{-(x + i\pi)} \]

\[ = e^{-(x + \pi)} \]

\[ = e^{-(\cos x + i\sin x)} \]

\[ \therefore 0 < c < 4 \] is condition.
2C-4

\[ a) \frac{\text{use } y' \text{ for } \frac{dy}{dx}, \ y \text{ for } \frac{dy}{dt}] \]

We have \( \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dx}{dt} \), \( x = e^t \), \( \frac{dx}{dt} = e^t \), \( \frac{dy}{dx} = e^t \)

\[ y' = y \cdot e^t \]
\[ y^4 = \frac{d}{dt}(y \cdot e^t) \cdot \frac{dt}{dx} \]
\[ = (y \cdot e^t \cdot y \cdot e^t) \cdot e^{-t} \]
\[ = (y \cdot y \cdot e^t) \cdot e^{-t} \]

Substituting into the ODE:
\[ x^2 y'' + p y' + q y = 0 \]
becomes
\[ (y' - y) + p y' + q y = 0 \]

\[ b) \ p = q = 1, \ \text{so we get } y' + y = 0, \ \text{whose solution is} \ y = c_1 \cos(x) + c_2 \sin(x) \]
\[ x = e^t \] gives \( y = c_1 \cos(\ln(x)) + c_2 \sin(\ln(x)) \)
\[ t = \ln(x) \]

2C-5

Char. eqn is \( Mr^2 + cr + k = 0 \)

For critical damping, it should have two equal roots; by quadratic formula
\[ r = \frac{-c \pm \sqrt{c^2 - 4mk}}{2M} \]
\[ \frac{c^2 - 4mk}{2M} = 0 \text{ is condition} \]

(when \( c^2 - 4mk < 0 \), get oscillations).

2C-6

\[ \text{Force triangle} \]
\[ F = ma \]
\[ mg \sin \alpha = \frac{mg \sin \alpha}{\alpha} \]
\[ \text{mgsin} \alpha - \text{meda} = \frac{mg \sin \alpha}{\alpha} \]
\[ \text{If } k \text{ small, } \sin \alpha \approx \alpha \]
\[ x' + x = 0 \]
\[ \text{If undamped, } c = 0, \ \text{get approx.} \]
\[ \frac{x'}{\alpha} + x = 0 \]

Sols are \( y = c_1 \cos(\frac{x}{\alpha}) + c_2 \sin(\frac{x}{\alpha}) \)

The period \( = \frac{2\pi}{\sqrt{\frac{g}{2}}} = 2\pi \sqrt{\frac{g}{2}} \)

(\( \text{so as length increases, so does the period; on the moon, it swings slower (wicker period)} \))

2C-7

\[ a) \ a + bx + ce^x \]
\[ b) a \cos 2x + b \sin 2x \]
\[ c) ax^2 e^x + b x^2 e^x \]
\[ d) \ ax^2 e^x \ (x \text{ is a double root of the eqn} \]
\[ e) ae^{-x} + bxe^{-x} \ (x \text{ is a root of the eqn)} \]
\[ f) \ (ax^2 + bx^2) e^{-x} \ (x \text{ is a double root of the eqn}) \]

2C-8

\[ b) y_a = a_1 \cos 2x + a_2 \sin 2x \]

To find \( y_a \), use under. coefficients:
\[ x^3 \text{ (mult. factor)} \]
\[ y'' = -c_1 \cos(x) - c_2 \sin(x) \]
\[ \text{and add:} \]
\[ 2 \cos x = 3c_1 \cos x + 3c_2 \sin x \]
\[ c_1 = \frac{2}{3}, \ c_2 = 0 \]

So \( y = c_1 \cos 2x + c_2 \sin 2x + \frac{2}{3} \cos x \)
\[ y_0 = 0 \Rightarrow a_1 + \frac{2}{3} = 0 \Rightarrow a_1 = -\frac{2}{3} \]
\[ y_0 = 1 \Rightarrow za_2 = 1 \Rightarrow a_2 = \frac{1}{2} \]

2C-9

\[ a) y_a = e^{x} + a_2 e^{-x}, \ \text{as usual} \]

Try \( y_a = c_1 e^{x} \)
\[ \text{mult. factor} \]
\[ x^3 \]
\[ y'' = e^{x}(x+1) \]
\[ \text{and add:} \]
\[ e^{x} = e^{x}(-\frac{6}{6} + 2c) + e^{x}(5c - \frac{6}{6} + c) \]
\[ c = -\frac{1}{4} \]

\[ \frac{q}{q} = a_1 e^x + a_2 e^{-x} - \frac{1}{4} x e^{-x} \]

\[ c) \ \text{Char eqn:} \ r^2 + r + 1 = 0, \ r = -1 \pm \frac{1}{2} \]
\[ \text{Try } y_a = e^{-x/2}(a_1 \cos \frac{\sqrt{3}}{2} x + a_2 \sin \frac{\sqrt{3}}{2} x) \]
\[ \text{Try } y_a = c_1 e^{x} + c_2 e^{-x} \]
\[ y'' = c_1 e^{x}(x+2) + c_2 e^{-x} \]
\[ 2x e^{x} = 3c_1 x e^{x} + (3c_1 + 3c_2) e^{x} \]
\[ c_1 = 2/3, \ c_2 = -2/3 \]
\[ y = e^{-x/2}(aq_1 \cos \frac{\sqrt{3}}{2} x + a_2 \sin \frac{\sqrt{3}}{2} x) \]
\[ + \frac{2}{3} \cdot e^{x}(x - 1) \]
2C-8

a) \( y_t = a_1 e^x + a_2 e^{-x} \)

Try: \( y_f = c_1 x + c_2 x + c_3 \)

\( y_f' = 2c_1 \)

\( x^2 = -c_1 x^2 + c_2 x + 2c_1 - c_3 \)

\( c_1 = -1, \ c_2 = 0, \ 2c_1 - c_3 = 0 \)

\( c_3 = -2 \)

\( y = a_1 e^x + a_2 e^{-x} - x^2 - 2 \)

\( y(0) = 0 \Rightarrow a_1 - a_2 - 2 = 0 \)

\( y'(0) = 0 \Rightarrow a_1 - a_2 = -1 \)

Solving,

\( a_1 = \frac{1}{2}, \ a_2 = \frac{3}{2} \)

\( y = \frac{1}{2} e^x + \frac{3}{2} e^{-x} - x^2 - 2 \)

2C-9

Write the ODE as \( Ly = g \)

Where \( L \) is the linear operator

\( L = D^2 + PD + q \)

By hypothesis,

\( Ly_1 = r_1 \)

\( Ly_2 = r_2 \)

Adding, \( L(y_1 + y_2) = r_1 + r_2 \)

(Using the linearity of \( L \):

\( L(y_1 + y_2) = Ly_1 + Ly_2 \))

\( y_1 + y_2 \) solves \( Ly = r_1 + r_2 \)

b) First consider \( y'' + 2y' + 2y = 2x \)

Try:

\( y_1 = c_1 x + c_2 \)

\( y_1' = c_1 \)

\( y_1'' = 0 \)

\( 2x = 2c_1 x + (2c_2 + 2c_1) \)

\( c_1 = 1, \ c_2 = -1 \)

\( y_1 = x - 1 \)

Then:

\( y'' + 2y' + 2y = \cos x \)

Try:

\( y_2 = a_1 \cos x + a_2 \sin x \)

\( y_2' = -a_1 \sin x + a_2 \cos x \)

\( y_2'' = -a_1 \cos x - a_2 \sin x \)

\( \cos x = \cos x (2a_1 + 2a_2 - a_1) + \sin x (2a_2 - 2a_1 - a_2) \)

\( a_1 + 2a_2 = 1 \)

\(-2a_1 + a_2 = 0 \)

\( a_1 = \frac{2}{5}, \ a_2 = \frac{3}{5} \)

\( y_2 = \frac{1}{5} a_2 x + \frac{3}{5} \sin x \)

2C-10

a) \( R = 0, \ E = 0 \)

Eqn is \( Lq'' + \frac{g}{C} = 0 \) or \( q'' + \frac{g}{LC} = 0 \)

Solving as usual,

\( g = \frac{C}{L} \cos \left( \frac{t}{\sqrt{LC}} \right) + \frac{C}{L} \sin \left( \frac{t}{\sqrt{LC}} \right) \)

Period is \( 2\pi \sqrt{LC} \) (in frequency)

\( f = \frac{1}{2\pi \sqrt{LC}} \)

b) Char. eqn is \( Lr^2 + Rr + \frac{1}{C} = 0 \)

Roots:

\( r = -R \pm \sqrt{R^2 - 4L/C} \)

Oscillates if \( R^2 - 4L/C < 0 \)

c) \( L \frac{d^2 q}{dt^2} + \frac{1}{C} = \omega_0 \cos \omega t \)

Sols of homog. eqn are

\( i = \frac{a_1}{\sqrt{LC}} \cos \frac{t}{\sqrt{LC}} + \frac{a_2}{\sqrt{LC}} \sin \frac{t}{\sqrt{LC}} \)

The particular soln \( y_p \)

will have from \( c_1 \cos \omega t + c_2 \sin \omega t \)

unless \( \omega = \frac{1}{\sqrt{LC}} \), in which case it will be \( c_1 \cos \omega t + c_2 \sin \omega t \),

which goes large as \( t \to \infty \).

Thus if \( \omega = \frac{1}{\sqrt{LC}} \), solns will be large in amplitude

\( \omega = \frac{1}{\sqrt{LC}} \)

- This is \( \omega_0 \)

The advantage of this method

(divide and conquer?) is that we don't have to assume

\( y_p = a_1 x + a_2 x + a_3 \cos x + a_4 \sin x \)

which would give 4 equations w/ 4 unknowns

to solve...

Using part (a), the solution is

\( y = y_1 + y_2 = x - 1 + \frac{1}{5} \cos x + \frac{3}{5} \sin x \)

Using part (a), the solution is

\( y = y_1 + y_2 = x - 1 + \frac{1}{5} \cos x + \frac{3}{5} \sin x \)
a) \[ y_a = C \cos x + D \sin x, \] as usual.

\[
W(y_1, y_2) = \begin{vmatrix}
\cos x & \sin x \\
-\sin x & \cos x
\end{vmatrix} = 1
\]

Let \( y_P = u_1 y_1 + u_2 y_2 \).

The equations for variation of pars. are:

\[
u'_1 \cos x + u'_2 \sin x = 0
\]

\[
u'_1 (-\sin x) + u'_2 \cos x = \tan x
\]

Either by elimination, or by Cramer's rule, we get as soln. (the det. is \( W(y_1, y_2) \))

\[
u'_1 = -\frac{y_2 f(x)}{W(y_1, y_2)} = -\sin x \tan x = \cos x \sec x
\]

\[
u'_2 = \frac{y_1 f(x)}{W(y_1, y_2)} = \cos x \tan x = \sin x
\]

e from tables

\[
u_1 = \sin x - \ln |\sec x + \tan x| \\
u_2 = \cos x
\]

\[
y_P = (\sin x - \ln |\sec x + \tan x|) \cos x
\]

\[
y_P = -\cos x (\ln |\sec x + \tan x|)
\]

b) Two indep. solns. of the assoc. homogen. eqns. are:

\[
y_1 = e^x, \quad y_2 = e^{-3x}
\]

\[
W(y_1, y_2) = -4e^{2x} = \begin{vmatrix}
e^x & e^{3x} \\
e^x & -e^{-3x}
\end{vmatrix}
\]

\[
y_P = u_1 y_1 + u_2 y_2
\]

The eqn. for variation of pars. are:

\[
u'_1 e^x + u'_2 e^{-3x} = 0
\]

\[
u'_1 e^x - u'_2 e^{-3x} = e^x
\]

Solve them by elimination, or by Cramer's rule. Following the latter, we get as soln. (e) :

\[
u'_1 = -\frac{y_2 f(x)}{W} = \frac{1}{4} e^{2x}
\]

\[
u'_2 = \frac{y_1 f(x)}{W} = -\frac{4e^{-2x}}{e^{-3x}} = -\frac{1}{4} e^{2x}
\]

\[
u_1 = -\frac{1}{4} e^{-2x}, \quad u_2 = -\frac{1}{8} e^{3x}
\]

and so:

\[
y_P = -\frac{1}{8} e^{-2x} e^x - \frac{1}{8} e^{2x} e^{-3x},
\]

or:

\[
y_P = -\frac{1}{4} e^{-x}
\]

c) Two indep. solns. of the assoc. homogen. eqn. are:

\[
y_1 = \cos 2x, \quad y_2 = \sin 2x
\]

(b) (by the formal method)

\[
W(y_1, y_2) = \begin{vmatrix}
\cos 2x & \sin 2x \\
-2 \sin 2x & 2 \cos 2x
\end{vmatrix} = 2
\]

Let \( y_P = u_1 y_1 + u_2 y_2 \).

Then:

\[
u'_1 \cos 2x + u'_2 \sin 2x = 0
\]

\[
u'_1 (-2 \sin 2x) + u'_2 (2 \cos 2x) = \sec^2 2x
\]

are the eqns. for the method of variation of pars. Solving them by elimination, or by Cramer's rule:

\[
u'_1 = -\frac{y_2 f(x)}{W} = -\sin 2x
\]

\[
u'_2 = \frac{y_1 f(x)}{W} = \cos 2x
\]

\[
u_1 = -\frac{1}{2} \frac{\cos 2x}{\cos^2 2x} + \frac{1}{2} \frac{\sin 2x}{\cos^2 2x}
\]

Integrating,

\[
u_1 = \frac{1}{2} \ln |\sec 2x + \tan 2x|
\]

\[
u_2 = \frac{1}{4} \ln |\sec x + \tan x| \cdot \sin 2x
\]

2D-2

\[
W(y_1, y_2) = \begin{vmatrix}
y_1 & y_2 \\
y'_1 & y'_2
\end{vmatrix} = -\frac{1}{x}, \quad \text{after some calculation.}
\]

\[
y_P = u_1 y_1 + u_2 y_2
\]

Equations for the method of variation of pars. are:

\[
u'_1 y_1 + u'_2 y_2 = 0
\]

(note: the ODE must be written)

\[
u'_1 y_1 + u'_2 y_2 = \frac{\cos x}{\sqrt{x}}
\]

Solving these by Cramer's rule:

\[
u'_1 = -\frac{y_2 f(x)}{W} = \cos x
\]

\[
u'_2 = \frac{y_1 f(x)}{W} = -\sin x \cos x
\]

\[
u_1 = \frac{x + \sin 2x}{x}, \quad u_2 = \frac{\cos 2x}{4}
\]

and so (using identities):

\[
y_P = \frac{x \sin x}{\sqrt{x}} + \frac{1}{2} \frac{\cos x}{\sqrt{x}} + \frac{1}{2} \frac{\sin x}{\sqrt{x}}
\]

so:

\[
y_P = \frac{x \sin x}{\sqrt{x}} + \frac{1}{2} \frac{\cos x}{\sqrt{x}}
\]

(The term \( \frac{1}{2} \frac{\cos x}{\sqrt{x}} \) is part of the general soln. \( y = \frac{y_P}{\sqrt{x}} \), so it can be omitted:

\[
y_P = \frac{\sqrt{x} \sin x}{2} \quad \text{is the best answer.}
\]
a) Let \( y_1, y_2 \) be solutions of the associated homogeneous equation:
\[
W = \begin{vmatrix}
 y_1(x) & y_2(x) \\
 y'_1(x) & y'_2(x)
\end{vmatrix}
\]
and the eqns for the method of variation of params are:
\[
 u'_1 y_1 + u'_2 y_2 = 0 \\
u'_1 y'_1 + u'_2 y'_2 = f(x)
\]
Solving by Cramer's rule gives:
\[
u_1' = \frac{-y_2(x)f(x)}{W(y_1(x), y_2(x))}, \quad u_2' = \frac{-y_1(x)f(x)}{W(y_1(x), y_2(x))}
\]
so that (we definite integrals so as to get a definite form)
\[
u_1(x) = \int \frac{-y_2(x)f(t)\, dt}{W(y_1(t), y_2(t))}, \quad u_2(x) = \int \frac{-y_1(x)f(t)\, dt}{W(y_1(t), y_2(t))}
\]

Thus:
\[
y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)
\]
we can put \( y_1(x) \) and \( y_2(x) \) inside the integral sign because they are "constants"—the integrations is with respect to \( t \), not \( x \) then we can add the integrands. The result is:
\[
y_p = \int \frac{-y_1(x)\, y_2'(t) + y_2(x)\, y_1'(t)\, f(t)\, dt}{W(y_1(t), y_2(t))}
\]

\[
y_p = \int \frac{-y_1(x)\, y_2(t) + y_2(x)\, y_1(t)\, f(t)\, dt}{W(y_1(t), y_2(t))}
\]

b) The arbitrary constants of integration—call them \( a_1 \) and \( a_2 \)—will change \( u_1 \) and \( u_2 \) by an additive constant:
\[
u_1 + a_1, \quad u_2 + a_2
\]
leading to the particular soln:
\[
y_p = (u_1 + a_1)y_1 + (u_2 + a_2)y_2
\]

\[
y_p = \frac{u_1y_1' + u_2y_2'}{W(y_1(t), y_2(t))} + a_1y_1 + a_2y_2
\]
The boxed part is the particular solution of part (a); the part added on is in the general soln \( y_0 \) to the associated homog. eqn, hence the particular soln is just as good a particular soln as the previous one.

It depends on the ODE form (it must be linear!) Undetermined coefficients require:

1. The ODE is linear, with constant coefficients.
2. The inhomogeneous term \( f(x) \) has a special form: a sum of terms of the form
   \[
   (\text{polynomial}) \cdot e^{ax}, \quad \{\sin bx, \cos bx\}
   \]
   
   \[\uparrow \quad \uparrow \quad \text{can be } 1, \text{any } b, \text{can be } 0\]

If the coeffs are not constant, or \( f(x) \) is not of the above form, you must use variation of parameters to find \( y_p \):

Drawback: you must be able to find \( y_1, y_2 \) first, i.e., solve the associated homog. eqn.

(Note that finding \( y_p \) by undet. coeffs does not require you to solve for \( y_1, y_2 \) first (unless you are unlucky and \( f(x) \) is a sum of the associated homog. eqn.)—but you can always test this without solving the eqn.)
2E-1 \[ 1+i = \sqrt{2} e^{i\pi/4} \]

2E-2 \[ \frac{1-i}{1+i} = \frac{(1-i)^2}{(1+i)(1-i)} = -\frac{2i}{2} = -i \]

Other way:

1 - i = \sqrt{2} e^{-i\pi/4}
1 + i = \sqrt{2} e^{i\pi/4}

\[ \frac{1-i}{1+i} = \frac{\sqrt{2} e^{-i\pi/4}}{\sqrt{2} e^{i\pi/4}} = e^{-i\pi/2} = -i \]

2E-4 \[ z = a+ib, \quad w = c+di \]

\[ zw = (ac-bd) + i(ad+bc) \]

\[ \overline{zw} = (ac-bd) - i(ad+bc) \]

\[ z \overline{w} = (a-bi)(c-di), \quad z \overline{w} = (ac-bd) - i(ad+bc) \]

2E-7 (a) \[ (1-i)^4 = 1 + 4(-i) + 6(-i)^2 + 4(-i)^3 \]

\[ = 1 - 6i + i(-4 - 8) = -4 \]

By DeMoivre:

\[ 1 - i = \sqrt{2} e^{-i\pi/4} \]

\[ (1-i)^4 = (\sqrt{2})^4 e^{-4i\pi/4} = 4 \cdot (-1) = -4 \]

b) \[ (1+i\sqrt{3})^3 = 1 + 3i\sqrt{3} + 3(\sqrt{3})^2 + (i\sqrt{3})^3 \]

\[ = 1 + 3\sqrt{3} + 9 - 3i\sqrt{3} = -8 + i(3\sqrt{3} - 3\sqrt{3}) = -8 \]

By polar form:

\[ 1+i\sqrt{3} = 2e^{i\pi/3} \]

\[ \frac{1+i\sqrt{3}}{\sqrt{3}} \]

2E-9 The sixth roots of 1 are \[ e^{\frac{2\pi i}{6}} \]

where \( k = 0, 1, 2, \ldots, 5 \)

\[ \text{set: } 1, -1, \frac{1 \pm i\sqrt{3}}{2} \]

2E-10 \[ \sqrt[4]{16} = 2 \cdot \sqrt[4]{-1} \]

The 4th roots of -1 are on the picture:

\[ \pm 1 \pm i \sqrt{2} \]

\[ \sqrt[4]{2} \left( \pm 1 \pm i \right) \text{ are the roots} \]

\[ 4 \cdot y^4 + 16 = 0 \]

2E-14 \[ \sin^4 x = \left( \frac{e^{ix} - e^{-ix}}{2i} \right)^4 \]

\[ = \frac{1}{16} \left( e^{4ix} - 4e^{2ix} - 4e^{-2ix} + e^{-4ix} \right) \]

\[ = \frac{1}{16} \left( e^{4ix} + e^{-4ix} \right) - \frac{1}{8} \left( e^{2ix} + e^{-2ix} \right) + \frac{1}{4} \]

\[ = \frac{1}{8} \cos 4x - \frac{1}{2} \cos 2x + \frac{3}{8} \]

Since \( \sin^4 x \) is an even function, the answer should not contain the odd functions \( \sin 4x, \sin 2x \).

2E-15 \[ e^{(2+i)x} = e^{2x} (\cos x + i \sin x) \]

So \( e^{2x} \sin x = \text{Im} \ e^{(2+i)x} \)

\[ \int e^{(2+i)x} \ dx = \frac{1}{2+i} e^{(2+i)x} \]

\[ \frac{2+i}{2} \left( e^{2x} \cos x + i e^{2x} \sin x \right) \]

We want just the imaginary part:

\[ \int e^{2x} \sin x \ dx = e^{2x} \left( \frac{2}{5} \sin x - \frac{1}{5} \cos x \right) \]

2E-16 \[ e^{ix} = \cos x + i \sin x \]

\[ e^{-ix} = \cos x - i \sin x \]

Adding: \[ e^{ix} + e^{-ix} = 2 \cos x \]

Subtract: \[ e^{ix} - e^{-ix} = 2 \sin x \]
a) \(x^2+2x+2 = 0\) has roots \(-1 \pm i\)

\[y = e^{2x}(c_1e^{x} + c_2e^{-x}) + e^{-x} \left( c_3\cos x + c_4\sin x \right)\]

b) \(D^3 - 2D + 1 = (D-1)^2(D+1)\)

\[= (D-1)^2(D+1)^2\]

\[= (D-1)^2(D+1)^2\]

\[y = e^x(c_1 + c_2x) + e^{-x}(c_3 + c_4x) + \cos x (c_5 + c_6x) + \sin x (c_7 + c_8x)\]

c) Characteristic eqn is \(z^2 + 1 = 0\)

Roots are \(i\) and \(-i\)

\[a = \frac{1+i}{\sqrt{2}}, \quad b = \frac{1-i}{\sqrt{2}}\]

\[y = e^{ax}(c_1\cos x + c_2\sin x) + e^{bx}(c_3\cos x + c_4\sin x)\]

d) Characteristic eqn is \(z^2 - 8z + 16 = 0\)

which factors as \((z-4)^2\) or \((z+2)^2\)

So the double root is \(2, -2\)

\[y = c_1e^{2x} + c_2xe^{2x} + c_3e^{-2x} + c_4xe^{-2x}\]

e) \(y = c_1e^{x} + c_2e^{-x}\)

\[+ e^{2x}(c_3\cos \frac{3\pi}{2}x + c_4\sin \frac{3\pi}{2}x)\]

\[+ e^{-2x}(c_3\cos \frac{5\pi}{2}x + c_4\sin \frac{5\pi}{2}x)\]

[using roots as given in soln to 2e-9]

f) \[y = e^{\sqrt{2}x}(c_1\cos \sqrt{2}x + c_2\sin \sqrt{2}x)\]

\[+ e^{-\sqrt{2}x}(c_3\cos \sqrt{2}x + c_4\sin \sqrt{2}x)\]

d) \[x^2 + 2x + 4 = 0; \quad x = -1 \pm \sqrt{-3} = -1 \pm i\sqrt{3}\]

Changing to polar representation: \(2e^{\pi/3}\), \(2e^{5\pi/3}\)

\[x = \sqrt{2}e^{\pi/4}; \quad \sqrt{2}e^{5\pi/4}\]

(Use rules of the first 7)

\[\sqrt{2}e^{3\pi/4}, \sqrt{2}e^{7\pi/4}\]

Using these and \(x^2 + 2x + 4 = 0\)

\[y = e^{\pi/4}(c_1\cos \frac{\pi}{4}x + c_2\sin \frac{\pi}{4}x)\]

\[+ e^{5\pi/4}(c_3\cos \frac{5\pi}{4}x + c_4\sin \frac{5\pi}{4}x)\]

leading to: \(y = e^{x}(c_1\cos \frac{3\pi}{4}x + c_2\sin \frac{3\pi}{4}x)\]

\[+ e^{-x}(c_3\cos \frac{5\pi}{4}x + c_4\sin \frac{5\pi}{4}x)\]
2F-4

\[ x''_1 + 2x_1 - x_2 = 0 \\
\]
\[ x''_2 + x_2 - x_1 = 0 \\
\]

Eliminate \( x_1 \) by solving for \( x_1 \):
\[ x_1 = x''_2 + x_2 \]

Substitute into the first equation:
\[ (x''_2 + x_2)^2 + 2(x''_2 + x_2)x_2 - x_2 = 0 \]

or
\[ x''_2 + 3x''_2 + x_2 = 0 \]

Char. eqn:
\[ z^2 + 3z^2 + 1 = 0 \]

as quadratic eqn \( z^2 \pm z^2 + 1 = 0 \):
\[ z^2 = -3 \pm \sqrt{9 - 4} \]

Both roots are real, + negative

\[ z = \frac{-3 \pm \sqrt{5}i}{2} \]

so
\[ X_2 = c_1 \cos at - c_2 \sin at + c_3 \cos bt + c_4 \sin bt \]

2F-5

\[ D^2e^{2x}e^{2x} = e^{2x}(D+2)^2e^{2x} \]

\[ = e^{2x}(D^2 + 4D + 4)e^{2x} \]

\[ = e^{2x}(3 + 4D + D^2)e^{2x} \]

2F-6

a) By (12) w notes, (see Example 2)
\[ y_p = \frac{4}{r+1} e^x = 2e^x \]

b) \((D^2 + D - 2)y) = 2e^{ix} \)
\[ \therefore y_p = 2e^{ix} \]

\[ y_p = 2 \left( \frac{1}{5} (e^{ix} + i e^{-ix}) \right) \]

Re \( y_p = \frac{1}{5} \cos x \]

(1+i)^2 = 2(1+i)
(1+i)^2 = 2
\[ \therefore y_p = \frac{e^{ix}}{2} \]

Re \( y_p = \frac{1}{2} e^{ix} \cos x \]

d) \((D^2 + 2D + 9) = (D-3)^2 \)
\[ \therefore y_p = c_1 e^{3x} e^{3x} \]

\((D-3)^2 y_p = 2c_1 e^{3x} \]

\( y_p = e^{3x} \) (Form the ODE)

\[ c = \frac{1}{2}, \quad y_p = \frac{1}{2} x^2 e^{3x} \]

2F-7

\((D+1)e^{ax}u = e^{ax}f(x) \)
\[ \therefore Du = e^{ax}f(x), \quad u = e^{ax} \int f(x) dx \]

\[ y_p = e^{-ax} e^{ax}f(x) dx \]

2G-1

\[ y'' + 2y' + cy = 0 \]

Char. eqn \( \lambda^2 + 2\lambda + c = 0 \)

By quadratic formula:
\[ \lambda = -1 \pm \sqrt{1-c} \]

Unstable \( \leftarrow \) Stable

2G-2

\[ \frac{r^2 + b}{a} + \frac{c}{a} = \left( r - r_1 \right) \left( r - r_2 \right) \]

\[ \therefore \frac{c}{a} = r_1 r_2 \]

Real case:
\[ r_1, r_2 < 0 \quad \Rightarrow \quad b/a > 0 \]

Complex case:
\[ r_1 = \alpha + \beta i, \quad \alpha < 0 \quad \Rightarrow \quad \frac{b}{a} = -2\alpha > 0 \]

2G-3

Assume \( a, b, c > 0 \) (if not, multiply DE through by \(-1\))
\[ \gamma = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

If roots are real, \( \frac{-b - \sqrt{b^2 - 4ac}}{2a} < 0 \)
and \( \frac{-b + \sqrt{b^2 - 4ac}}{2a} < 0 \), therefore (since \( b^2 - 4ac < b^2 \)).

If roots are complex, \( \frac{-b}{2a} < 0 \)

\[ \therefore \text{in both cases, the char. roots have negative real part.} \]
2H-1. \[ y'' - k^2 y = 0, \quad y(0) = 0, \quad y'(0) = 1 \]

\[ y_c = c_1 e^{kx} + c_2 e^{-kx} \]

Solution to IVP \( \omega \)

\[ W(t) = \frac{e^{kx} - e^{-kx}}{2k} = \frac{\sinh kx}{k} \]

2H-3a. By Example 2 (p. 2),

\[ w(x) = \frac{e}{x} e^{-2x} \]

Therefore

\[ y(x) = \int \frac{(x-t)e^{-2(x-t)} - e^{-2t} dt}{w(x-t) - f(t)} \]

\[ = \frac{e^{-2x} \int_0^x (x-t) dt}{\int_0^x (x-t) dt} = \frac{e^{-2x} (x^2 - \frac{x^2}{2})}{\frac{x^2}{2}} = \frac{x^2 e^{-2x}}{2} \]

By undetermined coefficients, since \( y = e^{-2x}(c_1 + c_2 x) \), try \( c_1 e^{-2x} \)

\( (D + 2)^2 \) \( c_1 e^{-2x} = c_1 e^{-2x} x^2 \]

\( c_1 e^{-2x} = c_1 e^{-2x} x^2 \)

Thus, the ODE, \( y'' - 2x \)

\( c_1 e^{-2x} x = e^{-2x} \); \( c = \frac{1}{2} \)

2H-4. a) By Leibniz:

\[ \phi'(x) = \frac{d}{dx} \int_0^x (2x+3t)^2 dt = \]

\[ = (2x)^2 + \int_0^x (2x+3t)^2 dt \]

\[ = (2x)^2 + 4(2x + \frac{3x^2}{2}) \int_0^x = (5x)^2 + 14x^2 \]

\[ \approx 39x^2 \]

b) Directly:

\[ \phi(x) = \frac{1}{q} (2x+3t)^2 \int_0^x = \frac{1}{q} (5x)^2 - (2x)^3 \]

So, \( \phi'(x) = 39x^3 \); \( \checkmark \)