3. Laplace Transform

3A. Elementary Properties and Formulas

3A-1. Show from the definition of Laplace transform that \( \mathcal{L}(t) = \frac{1}{s}, \ s > 0. \)

3A-2. Derive the formulas for \( \mathcal{L}(e^{at} \cos bt) \) and \( \mathcal{L}(e^{at} \sin bt) \) by assuming the formula
\[
\mathcal{L}(e^{at}) = \frac{1}{s - \alpha}
\]
is also valid when \( \alpha \) is a complex number; you will also need
\[
\mathcal{L}(u + iv) = \mathcal{L}(u) + i\mathcal{L}(v),
\]
for a complex-valued function \( u(t) + iv(t). \)

3A-3. Find \( \mathcal{L}^{-1}(F(s)) \) for each of the following, by using the Laplace transform formulas. (For (c) and (e) use a partial fractions decomposition.)

\[
a) \ \frac{1}{2s + 3} \quad b) \ \frac{3}{s^2 + 4} \quad c) \ \frac{1}{s^2 - 4} \quad d) \ \frac{1 + 2s}{s^3} \quad e) \ \frac{1}{s^3 - 9s^2}
\]

3A-4. Deduce the formula for \( \mathcal{L} \) \( (\sin at) \) from the definition of Laplace transform and the formula for \( \mathcal{L} \) \( (\cos at) \), by using integration by parts.

3A-5. a) Find \( \mathcal{L}(\cos^2 at) \) and \( \mathcal{L}(\sin^2 at) \) by using a trigonometric identity to change the form of each of these functions.

b) Check your answers to part (a) by calculating \( \mathcal{L}(\cos^2 at) + \mathcal{L}(\sin^2 at) \). By inspection, what should the answer be?

3A-6. a) Show that \( \mathcal{L} \left( \frac{1}{\sqrt{t}} \right) = \sqrt{\frac{\pi}{s}} \), \( s > 0 \), by using the well-known integral
\[
\int_{0}^{\infty} e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2}.
\]
(Hint: Write down the definition of the Laplace transform, and make a change of variable in the integral to make it look like the one just given. Throughout this change of variable, \( s \) behaves like a constant.)

b) Deduce from the above formula that \( \mathcal{L} \left( \sqrt{t} \right) = \frac{\sqrt{\pi}}{2s^{3/2}} \), \( s > 0. \)

3A-7. Prove that \( \mathcal{L}(e^{at}) \) does not exist for any interval of the form \( s > a. \)
(Show the definite integral does not converge for any value of \( s. \))

3A-8. For what values of \( k \) will \( \mathcal{L}(1/t^k) \) exist? (Write down the definition of this Laplace transform, and determine for what \( k \) it converges.)

3A-9. By using the table of formulas, find:
   a) \( \mathcal{L}(e^{-t} \sin 3t) \)  b) \( \mathcal{L}(e^{2t}(t^2 - 3t + 2)) \)

3A-10. Find \( \mathcal{L}^{-1}(F(s)) \), if \( F(s) = \)
   \[
a) \ \frac{3}{(s - 2)^4} \quad b) \ \frac{1}{s(s - 2)} \quad c) \ \frac{s + 1}{s^2 - 4s + 5}
\]
3B. Derivative Formulas; Solving ODE’s

3B-1. Solve the following IVP’s by using the Laplace transform:

a) \( y' - y = e^{3t}, \quad y(0) = 1 \)
   b) \( y'' - 3y' + 2y = 0, \quad y(0) = 1, \ y'(0) = 1 \)
   c) \( y'' + 4y = \sin t, \quad y(0) = 1, \ y'(0) = 0 \)
   d) \( y'' - 2y' + 2y = 2e^t, \quad y(0) = 0, \ y'(0) = 1 \)
   e) \( y'' - 2y' + y = e^t, \quad y(0) = 1, \ y'(0) = 0 \).

3B-2. Without referring to your book or to notes, derive the formula for \( \mathcal{L}(f'(t)) \) in terms of \( \mathcal{L}(f(t)) \). What are the assumptions on \( f(t) \) and \( f'(t) \)?

3B-3. Find the Laplace transforms of the following, using formulas and tables:

a) \( t \cos bt \)
   b) \( t^n e^{kt} \) (two ways)
   c) \( e^{at} t \sin t \)

3B-4. Find \( \mathcal{L}^{-1}(F(s)) \) if \( F(s) = \)

a) \( \frac{s}{(s^2 + 1)^2} \)
   b) \( \frac{1}{(s^2 + 1)^2} \)

3B-5. Without consulting your book or notes, derive the formulas

a) \( \mathcal{L}(e^{at} f(t)) = F(s - a) \)
   b) \( \mathcal{L}(t f(t)) = -F'(s) \)

3B-6. If \( y(t) \) is a solution to the IVP \( y'' + ty = 0, \ y(0) = 1, \ y'(0) = 0 \), what ODE is satisfied by the function \( Y(s) = \mathcal{L}(y(t)) \) ?

(The solution \( y(t) \) is called an Airy function; the ODE it satisfies is the Airy equation.)

3C. Discontinuous Functions

3C-1. Find the Laplace transforms of each of the following functions; do it as far as possible by expressing the functions in terms of known functions and using the tables, rather than by calculating from scratch. In each case, sketch the graph of \( f(t) \). (Use the unit step function \( u(t) \) wherever possible.)

a) \( f(t) = \begin{cases} 1, & 0 \leq t \leq 1 \\ -1, & 1 < t \leq 2 \\ 0, & \text{otherwise} \end{cases} \)
   b) \( f(t) = \begin{cases} t, & 0 \leq t \leq 1 \\ 2 - t, & 1 \leq t \leq 2 \\ 0, & \text{otherwise} \end{cases} \)
   c) \( f(t) = |\sin t|, \quad t \geq 0. \)

3C-2. Find \( \mathcal{L}^{-1} \) for the following:

a) \( \frac{e^{-s}}{s^2 + 3s + 2} \)
   b) \( \frac{e^{-s} - e^{-3s}}{s} \) (sketch answer)

3C-3. Find \( \mathcal{L}(f(t)) \) for the square wave \( f(t) = \begin{cases} 1, & 2n \leq t \leq 2n + 1, \quad n = 0, 1, 2, \ldots \\ 0, & \text{otherwise} \end{cases} \)

a) directly from the definition of Laplace transform;
   b) by expressing \( f(t) \) as the sum of an infinite series of functions, taking the Laplace transform of the series term-by-term, and then adding up the infinite series of Laplace transforms.

3C-4. Solve by the Laplace transform the following IVP, where \( h(t) = \begin{cases} 1, & \pi \leq t \leq 2\pi, \\ 0, & \text{otherwise} \end{cases} \)

\[ y'' + 2y' + 2y = h(t), \quad y(0) = 0, \ y'(0) = 1; \]
write the solution in the format used for $h(t)$.

3C-5. Solve the IVP: $y'' - 3y' + 2y = r(t)$, $y(0) = 1$, $y'(0) = 0$, where $r(t) = u(t)t$, the ramp function.

3D. Convolution and Delta Function

3D-1. Solve the IVP: $y'' + 2y' + y = \delta(t) + u(t-1)$, $y(0) = 0$, $y'(0^-) = 1$.

Write the answer in the “cases” format $y(t) = \begin{cases} \cdot \cdot \cdot , & 0 \leq t \leq 1 \\ \cdot \cdot \cdot , & t > 1 \end{cases}$

3D-2. Solve the IVP: $y'' + y = r(t)$, $y(0) = 0$, $y'(0) = 1$, where $r(t) = \begin{cases} 1, & 0 \leq x \leq \pi \\ 0, & \text{otherwise} \end{cases}$

Write the answer in the “cases” format (see 3D-1 above).

3D-3. If $f(t + c) = f(t)$ for all $t$, where $c$ is a fixed positive constant, the function $f(t)$ is said to be periodic, with period $c$. (For example, $\sin x$ is periodic, with period $2\pi$.)

a) Show that if $f(t)$ is periodic with period $c$, then its Laplace transform is

$$F(s) = \frac{1}{1 - e^{-cs}} \int_0^c e^{-st} f(t) \, dt.$$ 

b) Do Exercise 3C-3, using the above formula.

3D-4. Find $\mathcal{L}^{-1}$ by using the convolution:

a) $\frac{s}{(s + 1)(s^2 + 4)}$

b) $\frac{1}{(s^2 + 1)^2}$

Your answer should not contain the convolution $\ast$.

3D-5. Assume $f(t) = 0$, for $t \leq 0$. Show informally that $\delta(t) \ast f(t) = f(t)$, by using the definition of convolution; then do it by using the definition of $\delta(t)$.

(See (5), section 4.6 of your book; $\delta(t)$ is written $\delta_0(t)$ there.)

3D-6. Prove that $f(t) \ast g(t) = g(t) \ast f(t)$ directly from the definition of convolution, by making a change of variable in the convolution integral.

3D-7. Show that the IVP: $y'' + k^2y = r(t)$, $y(0) = 0$, $y'(0) = 0$ has the solution

$$y(t) = \frac{1}{k} \int_0^t r(u) \sin k(t - u) \, du,$$

by using the Laplace transform and the convolution.

3D-8. By using the Laplace transform and the convolution, show that in general the IVP (here $a$ and $b$ are constants):

$$y'' + ay' + by = r(t), \quad y(0) = 0, \quad y'(0) = 0,$$

has the solution

$$y(t) = \int_0^t w(t - u)r(u) \, du,$$

where $w(t)$ is the solution to the IVP: $y'' + ay' + by = 0$, $y(0) = 0$, $y'(0) = 1$.

(The function $w(t - u)$ is called the **Green’s function** for the linear operator $D^2 + aD + b$.)