13. Natural frequency and damping ratio

There is a standard, and useful, normalization of the second order homogeneous linear constant coefficient ODE

\[ m\ddot{x} + b\dot{x} + kx = 0 \]

under the assumption that both the “mass” \( m \) and the “spring constant” \( k \) are positive. It is illustrated in the Mathlet Damping Ratio.

In the absence of a damping term, the ratio \( k/m \) would be the square of the circular frequency of a solution, so we will write \( k/m = \omega_n^2 \) with \( \omega_n > 0 \), and call \( \omega_n \) the natural circular frequency of the system.

Divide the equation through by \( m \): \( \ddot{x} + (b/m)\dot{x} + \omega_n^2 x = 0 \). Critical damping occurs when the coefficient of \( \dot{x} \) is \( 2\omega_n \). The damping ratio \( \zeta \) is the ratio of \( b/m \) to the critical damping constant: \( \zeta = (b/m)/(2\omega_n) \). The ODE then has the form

\[ \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0 \]  

(1)

Note that if \( x \) has dimensions of cm and \( t \) of sec, then \( \omega_n \) had dimensions sec\(^{-1}\), and the damping ratio \( \zeta \) is “dimensionless,” a number which is the same no matter what units of distance or time are chosen. Critical damping occurs precisely when \( \zeta = 1 \): then the characteristic polynomial has a repeated root: \( p(s) = (s + \omega_n)^2 \).

In general the characteristic polynomial is \( s^2 + 2\zeta\omega_n s + \omega_n^2 \), and it has as roots

\[ -\zeta\omega_n \pm \sqrt{\zeta^2\omega_n^2 - \omega_n^2} = \omega_n(-\zeta \pm \sqrt{\zeta^2 - 1}). \]

These are real when \( |\zeta| \geq 1 \), equal when \( \zeta = \pm 1 \), and nonreal when \( |\zeta| < 1 \). When \( |\zeta| \leq 1 \), the roots are

\[ -\zeta\omega_n \pm i\omega_d \]

where

\[ \omega_d = \sqrt{1 - \zeta^2} \omega_n \]  

(2)

is the damped circular frequency of the system. These are complex numbers of magnitude \( \omega_n \) and argument \( \pm \theta \), where \( -\zeta = \cos \theta \). Note that the presence of a damping term decreases the frequency of a solution to the undamped equation—the natural frequency \( \omega_n \)—by the factor \( \sqrt{1 - \zeta^2} \). The general solution is

\[ x = Ae^{-\zeta\omega_n t} \cos(\omega_d t - \phi) \]  

(3)
Suppose we have such a system, but don’t know the values of \( \omega_n \) or \( \zeta \). At least when the system is underdamped, we can discover them by a couple of simple measurements of the system response. Let’s displace the mass and watch it vibrate freely. If the mass oscillates, we are in the underdamped case. We can find \( \omega_d \) by measuring the times at which \( x \) achieves its maxima. These occur when the derivative vanishes, and
\[
\dot{x} = A e^{-\zeta \omega_n t} (-\zeta \omega_n \cos(\omega_d t - \phi) - \omega_d \sin(\omega_d t - \phi)).
\]
The factor in parentheses is sinusoidal with circular frequency \( \omega_d \), so successive zeros are separated from each other by a time lapse of \( \pi / \omega_d \). If \( t_1 \) and \( t_2 \) are the times of neighboring maxima of \( x \) (which occur at every other extremum) then \( t_2 - t_1 = 2\pi / \omega_d \), so we have discovered the damped natural frequency:
\[
(4) \quad \omega_d = \frac{2\pi}{t_2 - t_1}.
\]
We can also measure the ratio of the value of \( x \) at two successive maxima. Write \( x_1 = x(t_1) \) and \( x_2 = x(t_2) \). The difference of their natural logarithms is the logarithmic decrement:
\[
\Delta = \ln x_1 - \ln x_2 = \ln \left( \frac{x_1}{x_2} \right).
\]
Then
\[
x_2 = e^{-\Delta} x_1.
\]
The logarithmic decrement turns out to depend only on the damping ratio, and to determine the damping ratio. To see this, note that the values of \( \cos(\omega_d t - \phi) \) at two points of time differing by \( 2\pi / \omega_d \) are equal. Using (3) we find
\[
\frac{x_1}{x_2} = e^{-\zeta \omega_n t} e^{-\zeta \omega_n (t_2 - t_1)} = e^{-2\zeta \omega_n t_1}.
\]
Thus, using (4) and (2),
\[
\Delta = \ln \left( \frac{x_1}{x_2} \right) = \zeta \omega_n (t_2 - t_1) = \zeta \omega_n \frac{2\pi}{\omega_d} = \frac{2\pi \zeta}{\sqrt{1 - \zeta^2}}.
\]
From the quantities \( \omega_d \) and \( \Delta \), which are directly measurable characteristics of the unforced system response, we can calculate the system parameters \( \omega_n \) and \( \zeta \):
\[
(5) \quad \zeta = \frac{\Delta/2\pi}{\sqrt{1 + (\Delta/2\pi)^2}}, \quad \omega_n = \frac{\omega_d}{\sqrt{1 - \zeta^2}} = \sqrt{1 + \left( \frac{\Delta}{2\pi} \right)^2} \omega_d.
\]