9. Normalization of solutions

9.1. Initial conditions. The general solution of any homogeneous linear second order ODE

\[ \ddot{x} + p(t)\dot{x} + q(t)x = 0 \]  

has the form \( c_1 x_1 + c_2 x_2 \), where \( c_1 \) and \( c_2 \) are constants. The solutions \( x_1, x_2 \) are often called “basic,” but this is a poorly chosen name since it is important to understand that there is absolutely nothing special about the solutions \( x_1, x_2 \) in this formula, beyond the fact that neither is a multiple of the other.

For example, the ODE \( \ddot{x} = 0 \) has general solution \( at + b \). We can take \( x_1 = t \) and \( x_2 = 1 \) as basic solutions, and have a tendency to do this or else list them in the reverse order, so \( x_1 = 1 \) and \( x_2 = t \). But equally well we could take a pretty randomly chosen pair of polynomials of degree at most one, such as \( x_1 = 4 + t \) and \( x_2 = 3 - 2t \), as basic solutions. This is because for any choice of \( a \) and \( b \) we can solve for \( c_1 \) and \( c_2 \) in \( at + b = c_1 x_1 + c_2 x_2 \). The only requirement is that neither solution is a multiple of the other. This condition is expressed by saying that the pair \( \{ x_1, x_2 \} \) is linearly independent.

Given a basic pair of solutions, \( x_1, x_2 \), there is a solution of the initial value problem with \( x(t_0) = a, \dot{x}(t_0) = b \), of the form \( x = c_1 x_1 + c_2 x_2 \). The constants \( c_1 \) and \( c_2 \) satisfy

\[
\begin{align*}
    a &= x(t_0) = c_1 x_1(t_0) + c_2 x_2(t_0) \\
    b &= \dot{x}(t_0) = c_1 \dot{x}_1(t_0) + c_2 \dot{x}_2(t_0).
\end{align*}
\]

For instance, the ODE \( \ddot{x} - x = 0 \) has exponential solutions \( e^t \) and \( e^{-t} \), which we can take as \( x_1, x_2 \). The initial conditions \( x(0) = 2, \dot{x}(0) = 4 \) then lead to the solution \( x = c_1 e^t + c_2 e^{-t} \) as long as \( c_1, c_2 \) satisfy

\[
\begin{align*}
    2 &= x(0) = c_1 e^0 + c_2 e^{-0} = c_1 + c_2, \\
    4 &= \dot{x}(0) = c_1 e^0 + c_2 (-e^{-0}) = c_1 - c_2,
\end{align*}
\]

This pair of linear equations has the solution \( c_1 = 3, c_2 = -1 \), so \( x = 3e^t - e^{-t} \).

9.2. Normalized solutions. Very often you will have to solve the same differential equation subject to several different initial conditions. It turns out that one can solve for just two well chosen initial conditions, and then the solution to any other IVP is instantly available. Here’s how.
Definition 9.2.1. A pair of solutions \( x_1, x_2 \) of (1) is \textit{normalized at} \( t_0 \) if
\[
\begin{align*}
x_1(t_0) &= 1, & x_2(t_0) &= 0, \\
\dot{x}_1(t_0) &= 0, & \dot{x}_2(t_0) &= 1.
\end{align*}
\]

By existence and uniqueness of solutions with given initial conditions, there is always exactly one pair of solutions which is normalized at \( t_0 \).

For example, the solutions of \( \ddot{x} = 0 \) which are normalized at 0 are \( x_1 = 1, x_2 = t \). To normalize at \( t_0 = 1 \), we must find polynomials of the form \( at + b \) with the right values and derivatives at \( t = 1 \). These are \( x_1 = 1, x_2 = t - 1 \).

For another example, the “harmonic oscillator”
\[
\ddot{x} + \omega_n^2 x = 0
\]
has basic solutions \( \cos(\omega_n t) \) and \( \sin(\omega_n t) \). They are normalized at 0 only if \( \omega_n = 1 \), since \( \frac{d}{dt} \sin(\omega_n t) = \omega_n \cos(\omega_n t) \) has value \( \omega_n \) at \( t = 0 \), rather than value 1. We can fix this (as long as \( \omega_n \neq 0 \)) by dividing by \( \omega_n \): so
\[
\cos(\omega_n t), \quad \omega_n^{-1} \sin(\omega_n t)
\]
is the pair of solutions to \( \ddot{x} + \omega_n^2 x = 0 \) which is normalized at \( t_0 = 0 \).

Here is another example. The equation \( \ddot{x} - x = 0 \) has linearly independent solutions \( e^t, e^{-t} \), but these are not normalized at any \( t_0 \) (for example because neither is ever zero). To find \( x_1 \) in a pair of solutions normalized at \( t_0 = 0 \), we take \( x_1 = ae^t + be^{-t} \) and find \( a, b \) such that \( x_1(0) = 1 \) and \( \dot{x}_1(0) = 0 \). Since \( \dot{x}_1 = ae^t - be^{-t} \), this leads to the pair of equations \( a + b = 1, a - b = 0 \), with solution \( a = b = 1/2 \). To find \( x_2 = ae^t + be^{-t} \ x_2(0) = 0, \dot{x}_2(0) = 1 \) imply \( a + b = 0, a - b = 1 \) or \( a = 1/2, b = -1/2 \). Thus our normalized solutions \( x_1 \) and \( x_2 \) are the \textit{hyperbolic sine} and \textit{cosine} functions:
\[
cosh t = \frac{e^t + e^{-t}}{2}, \quad \sinh t = \frac{e^t - e^{-t}}{2}.
\]

These functions are important precisely because they occur as normalized solutions of \( \ddot{x} - x = 0 \).

Normalized solutions are always linearly independent: \( x_1 \) can’t be a multiple of \( x_2 \) because \( x_1(t_0) \neq 0 \) while \( x_2(t_0) = 0 \), and \( x_2 \) can’t be a multiple of \( x_1 \) because \( \dot{x}_2(t_0) \neq 0 \) while \( \dot{x}_1(t_0) = 0 \).

Now suppose we wish to solve (1) with the general initial conditions.
If $x_1$ and $x_2$ are a pair of solutions normalized at $t_0$, then the solution $x$ with $x(t_0) = a$, $\dot{x}(t_0) = b$ is

$$x = ax_1 + bx_2.$$ 

The constants of integration are the initial conditions.

If I want $x$ such that $\ddot{x} + x = 0$ and $x(0) = 3$, $\dot{x}(0) = 2$, for example, we have $x = 3\cos t + 2\sin t$. Or, for an other example, the solution of $\ddot{x} - x = 0$ for which $x(0) = 2$ and $\dot{x}(0) = 4$ is $x = 2\cosh(t) + 4\sinh(t)$. You can check that this is the same as the solution given above.

**Exercise 9.2.2.** Check the identity

$$\cosh^2 t - \sinh^2 t = 1.$$  

9.3. **ZSR and ZIR.** There is an interesting way to decompose the solution of a linear initial value problem which is appropriate to the *inhomogeneous* case and which arises in the system/signal approach. Two distinguishable bits of data determine the choice of solution: the initial condition, and the input signal.

Suppose we are studying the initial value problem

(3) \[ \ddot{x} + p(t)\dot{x} + q(t)x = f(t), \quad x(t_0) = x_0, \quad \dot{x}(t_0) = \dot{x}_0. \]

There are two related initial value problems to consider:

[ZSR] The same ODE but with rest initial conditions (or “zero state”):

\[ \ddot{x} + p(t)\dot{x} + q(t)x = f(t), \quad x(t_0) = 0, \quad \dot{x}(t_0) = 0. \]

Its solution is called the **Zero State Response**, or ZSR. It depends entirely on the input signal, and assumes zero initial conditions. We’ll write $x_f$ for it, using the notation for the input signal as subscript.

[ZIR] The associated homogeneous ODE with the given initial conditions:

\[ \ddot{x} + p(t)\dot{x} + q(t)x = 0, \quad x(t_0) = x_0, \quad \dot{x}(t_0) = \dot{x}_0. \]

Its solution is called the **Zero Input Response**, or ZIR. It depends entirely on the initial conditions, and assumes null input signal. We’ll write $x_h$ for it, where $h$ indicates “homogeneous.”

By the superposition principle, the solution to (3) is precisely

$$x = x_f + x_h.$$ 

The solution to the initial value problem (3) is the sum of a ZSR and a ZIR, in exactly one way.
Example 9.3.1. Drive a harmonic oscillator with a sinusoidal signal:
\[ \ddot{x} + \omega_n^2 x = a \cos(\omega t) \]
(so \( f(t) = a \cos(\omega t) \)) and specify initial conditions \( x(0) = x_0, \dot{x}(0) = \dot{x}_0 \). Assume that the system is not in resonance with the signal, so \( \omega \neq \omega_n \). Then the Exponential Response Formula (Section 10) shows that the general solution is
\[ x = a \frac{\cos(\omega t)}{\omega_n^2 - \omega^2} + b \cos(\omega_n t) + c \sin(\omega_n t) \]
where \( b \) and \( c \) are constants of integration. To find the ZSR we need to find \( \dot{x} \), and then arrange the constants of integration so that both \( x(0) = 0 \) and \( \dot{x}(0) = 0 \). Differentiate to see
\[ \dot{x} = -a \frac{\omega \sin(\omega t)}{\omega_n^2 - \omega^2} - b \omega_n \sin(\omega_n t) + c \omega_n \cos(\omega_n t) \]
so \( \dot{x}(0) = c \omega_n \), which can be made zero by setting \( c = 0 \). Then \( x(0) = a/(\omega_n^2 - \omega^2) + b \), so \( b = -a/(\omega_n^2 - \omega^2) \), and the ZSR is
\[ x_f = a \frac{\cos(\omega t) - \cos(\omega_n t)}{\omega_n^2 - \omega^2} . \]

The ZIR is
\[ x_h = b \cos(\omega_n t) + c \sin(\omega_n t) \]
where this time \( b \) and \( c \) are chosen so that \( x_h(0) = x_0 \) and \( \dot{x}_h(0) = \dot{x}_0 \). Thus (using (2) above)
\[ x_h = x_0 \cos(\omega_n t) + \dot{x}_0 \frac{\sin(\omega_n t)}{\omega_n} . \]

Example 9.3.2. The same works for linear equations of any order. For example, the solution to the bank account equation (Section 2)
\[ \dot{x} - I x = c , \quad x(0) = x_0 , \]
(where we’ll take the interest rate \( I \) and the rate of deposit \( c \) to be constant, and \( t_0 = 0 \)) can be written as
\[ x = \frac{c}{I} (e^{It} - 1) + x_0 e^{It} . \]
The first term is the ZSR, depending on \( c \) and taking the value 0 at \( t = 0 \). The second term is the ZIR, a solution to the homogeneous equation depending solely on \( x_0 \).