18.034 Problem Set #4

Due by Friday, March 13, 2009, by NOON.

1. (a) Show that if $u_1$ is a solution of the third-order variable-coefficient linear differential equation
   \[ u'''' + p_1(x)u'' + p_2(x)u' + p_3(x)u = 0, \]
   then the substitution $u(x) = u_1(x)v(x)$ leads to the second-order differential equation for $v'$:
   \[ u_1v'''' + (3u'_1 + p_1u_1)v'' + (3u''_1 + 2p_1u'_1 + p_2u_1)v' = 0. \]
   (b) Verify that $u_1(x) = e^x$ is a solution of the differential equation $(2-x)u'''' + (2x-3)u'' - xu' + u = 0$. Use the method in part (a) to find the general solution of the differential equation.

2. Find a particular solution of the differential equation
   \[ x^2u'' + (1 - \alpha - \beta)xu' + \alpha\beta u = x^2f(x) \]
   (a) when $\alpha \neq \beta$ and (b) when $\alpha = \beta$.


4. (a) Find annihilators of $x^m e^{ax}$, $x^m \sin \beta x$, and $e^{ax} \cos \beta x$.
   (b) Find the general solution of $(D^2 + 1)(D - 3)^2u = 12e^{3x}(10x + 1)$.

5. Consider the $n$th order linear homogeneous differential equation
   \[ u^{(n)} + a_1u^{(n-1)} + \cdots + a_{n-1}u' + a_nu = 0 \]
   with real constant coefficients.
   (a) Show that if the differential equation is asymptotically stable then $a_1, a_2, \ldots, a_n > 0$.
   (b) Show that the converse to part (a) is true if all roots of its associated characteristic polynomial are real. (Hint. Assume it is false, and seek for a contradiction.)
   (c) Show by a counter example that the converse to part (a) is in general false.

6. (a) Show that the function $y^2$ is not Lipschitzian on $-\infty < y < \infty$. Discuss how this failure of the Lipschitz condition is reflected in the behavior of the initial value problem
   \[ y' = y^2, \quad y(0) = y_0 > 0. \]
   (b) Show that the function $y^{2/3}$ is not Lipschitzian in any strip $|y| < h$ containing the origin. Discuss how this failure of the Lipschitz condition is reflected in the behavior of the initial value problem
   \[ y' = y^{2/3}, \quad y(0) = 0. \]