18.034 Honors Differential Equations
Spring 2009

For information about citing these materials or our Terms of Use, visit: http://ocw.mit.edu/terms.
18.034 Problem Set #6
(modified on April 8, 2009)

Due by Friday, April 10, 2009, by NOON.

1. (Laplace transform of $t^r$). The Gamma function is defined by the integral
\[
\Gamma(r + 1) = \int_0^\infty e^{-t^r} dt.
\]
(a) Show that the improper integral converges for all $r > -1$.
(b) Show that $\Gamma(r + 1) = r\Gamma(r)$ for $r > 0$. Show that $\Gamma(1) = 1$, $\Gamma(1/2) = \sqrt{\pi}$.
(c) For $r > -1$ show that $\mathcal{L}[t^r] = \Gamma(r + 1)/s^{r+1}, s > 0$

2. (a) Find the solution of the initial value problem
\[
y'' + \omega^2 y = h(t) \sin t - h(t - c) \sin t, \quad y(0) = y'(0) = 0,
\]
where $c > 0$ is a constant and $\omega^2 \neq 1$.
(b) Show that $y(0) = y'(0) = y''(0) = 0$. Show that $y$ and $y'$ are continuous at $t = c$.
(c) Show that $y''(c+) - y''(c-) = -\sin c$, which is 0 if and only if $c = n\pi$ for $n$ an integer. This behavior is explained by that the function $h(t - c) \sin t$ is continuous at $c$ if and only if $\sin c = 0$.

3. Find the Laplace transform of a full rectified wave $^* f(t) = |\sin t|$.

4. Find the solution of the initial value problem
\[
y'' + y = A\delta(t - c), \quad y(0) = a, \quad y'(0) = b,
\]
where $A, a, b, c$ are constant and $c > 0$.
(b) Show that $y(t) = 0$ for $t \geq c$ if and only if $y(c) = 0$ and
\[
A = \sqrt{a^2 + b^2}, \quad a \sin c \geq b \cos c; \quad A = -\sqrt{a^2 + b^2}, \quad a \sin c \leq b \cos c.
\]
To interpret, these amplitudes $A$ and locations of impulse $c$ cancel the oscillation.

5. (The Volterra integral equation). Consider the integral equation
\[
y(t) + \int_0^t (t - s)y(s) ds = -\frac{1}{4} \sin 2t.
\]
(a) Show that the above integral equation is equivalent to the initial value problem
\[
y'' + y = \sin 2t, \quad y(0) = 0, \quad y'(0) = -\frac{1}{2}.
\]
(b) Solve the integral equation by using the Laplace transform.

*It describes a direct current.
6. Consider the Bessel equation of order zero

\[ ty'' + y' + ty = 0. \]

Note that \( t = 0 \) is a singular point and thus solutions may become unbounded as \( t \to 0 \). Nevertheless, let us try to determine whether there are any solutions that remain finite at \( t = 0 \) and have finite derivatives there.

(a) Show that \( Y(s) = \mathcal{L}[y](s) \) satisfies \( (1 + s^2)Y''(s) + sY(s) = 0 \).

(b) Using the binomial series for \((1 + s^2)^{-1/2}\) for \( s > 1 \) show that

\[
y(t) = c \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2^{2n} (n!)^2},
\]

which is referred to as the Bessel function of the first kind of order zero. Show that \( y(0) = 1 \) and \( y \) has finite derivatives for all orders at \( t = 0 \).