18.034 Problem Set #7
(modified on April 13, 2009)

Due by Friday, April 17, 2009, by NOON.

1. In many applications the input \( f \) and the output \( y \) are related by \( L_2y = L_1f \), where \( L_1 \) and \( L_2 \) are linear differential operators.
   (a) If \( P_2 \) is the characteristic polynomial of \( L_j, j = 1, 2 \), show that
   \[
   P_2(s)Y(s) = P_1(s)F(s) + P_0(s),
   \]
   where \( P_0(s) \) is a polynomial depending on the initial conditions of both \( f \) and \( y \).
   (b) The null initial condition gives the transfer function \( W(s) = P_1(s)/P_2(s) \). If \( W(s) = \mathcal{L}w \), express \( y \) by the convolution theorem.
   (c) We can have a function \( w \) as in part (b) only if the degree of \( P_1 \) is less than that of \( P_2 \). Why?

2. One method of determining the ratio \( e/m \) of charge to mass for an electron leads to the system
   \[
   mx'' + Hey' = eE, \quad my'' - Hex' = 0.
   \]
   The initial values are \( x(0) = x'(0) = y(0) = y'(0) = 0 \) and \( m, e, H, E \) are positive constants.
   (a) Solve by the Laplace transform and show that the path of a particle in the coordinates \( (x, y) \) are given by
   \[
   x(t) = E(\omega H)^{-1}(1 - \cos \omega t), \quad y(t) = E(\omega H)^{-1}(\omega t - \sin \omega t),
   \]
   where \( \omega = He/m \). This is a cycloid generated by a circle of radius \( E(\omega H)^{-1} \).
   (b) How can one determine the system-parameter \( e/m \) from knowledge of the path and the values \( E \) and \( H \) set by the experimenter?

3. Consider the system of differential equations \( Y' = AY \), where
   \[
   Y = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}, \quad A = \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix}.
   \]
   (a) If \( U(t) \) is a fundamental solution and \( C \) is a nonsingular matrix, show that \( UC \) is also a fundamental solution.
   (b) If \( V(t) \) is a fundamental solution, show that \( U(t) = V(t)V(t_0)^{-1} \) satisfies \( U(t_0) = I \) and is also a fundamental solution.
   (c) Show that \( |Y'| = D_1 + D_2 \), where
   \[
   D_1 = \begin{vmatrix} a_{11}y_{11} + a_{12}y_{21} & a_{11}y_{12} + a_{12}y_{22} \\ y_{21} & y_{22} \end{vmatrix} = a_{11}|Y|
   \]
   and \( D_2 = a_{22}|Y'| \). Deduce the Liouville theorem.

4. (a) Show that if \( (y_1(t), y_2(t)) \) is a solution of
   \[
   t^2y_1' = -2ty_1 + 4y_2, \quad ty_2' = -2ty_1 + 5y_2
   \]
   then both \( y_1 \) and \( y_2 \) are solutions of the Euler equation
   \[
   (t^2\phi' + 2t\phi) = -8\phi + 5(t\phi' + 2\phi).
   \]
(b) Find a fundamental matrix for
\[
\begin{pmatrix}
-2t^{-1} & 4t^{-2} \\
-2 & 5t^{-1}
\end{pmatrix}
\]

(c) For \( A = \frac{1}{t^2} \begin{pmatrix} 2t & -2 \\ t^2 + 2t & -2 \end{pmatrix} \) obtain a fundamental matrix.

5. Solve the initial value problem
\[
\frac{d}{dt} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}
\]
(a) by using the eigenvalues and (b) by using the Laplace transform.

(c) For which values of \( a \) and \( b \), the solution of the initial value problem
\[
\frac{d}{dt} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}
\]
has the behavior \( \lim_{t \to \infty} (y_1(t), y_2(t))^T = (0, 0) \)?

6. For which vector \((a, b, c)\), is the solution of the initial value problem
\[
\frac{d}{dt} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \quad \begin{pmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}
\]
periodic in \( t \)?