18.034 Problem Set #8  
(modified on April 27, 2009)

Due by Friday, May 1, 2009, by NOON.

1. An $n \times n$ complex matrix $A$ is called Hermitian if $A = A^*$, where $A^*$ is the conjugate transpose of $A$. That is, $a_{ij} = \bar{a}_{ji}$ for all $1 \leq i, j \leq n$. When $A$ is real $A^* = A^T$ and the terms “Hermitian” and "symmetric" mean the same thing. If $\vec{u} = (u_1, \ldots, u_n)$ and $\vec{v} = (v_1, \ldots, v_n)$ are column vectors in $\mathbb{R}^n$, then $\vec{u} \vec{v}^* = u_1 \bar{v}_1 + \cdots + u_n \bar{v}_n$ and $\|u\|^2 = uu^*$.

   If $A = A^*$ show that all eigenvalues of $A$ are real. Furthermore, if $A = A^*$ then eigenvectors $\vec{u}$ and $\vec{v}$ corresponding to different eigenvalues $\lambda$ and $\mu$ are orthogonal. That is, $\vec{u} \vec{v}^* = 0$.

2. If $A$ and $B$ are $n \times n$ matrices, compute
   \[ \lim_{t \to 0} \frac{e^{At}e^{Bt} - e^{Bt}e^{At}}{t^2}. \]

3. (a) Let $A$ be a $3 \times 3$ matrix with eigenvalues $\lambda_1, \lambda_2, \lambda_3$. Show that the nonzero columns of $(A - \lambda_2 I)(A - \lambda_3 I)$ are eigenvectors for $\lambda_1$.
   (b) A $3 \times 3$ matrix $A$ has characteristic polynomial $p(\lambda) = \lambda(\lambda^2 - 1)$. Find $e^{At}$.

4. (a) For $x' = 6x + y, y' = 4x + 3y$ show that the origin is an unstable node.
   (b) If $y = mx$ is a trajectory, show that $m = 1$ or $m = -4$.
   (c) Sketch the trajectories in the $(x, y)$-plane.

5. Repeat Problem 4 for $x' = -3x + 2y, y' = -3x + 4y$.

6. Consider the differential equation $u'' + p(t)u' + q(t)u = 0$, where $p(t), q(t)$ are continuous functions on some interval of $t$.
   (a) Let
   \[ u(t) = r(t) \sin \theta(t), \quad u'(t) = r(t) \cos \theta(t). \]
   Show that
   \[ \frac{d\theta}{dt} = \cos^2 \theta + p(t) \cos \theta \sin \theta + q(t) \sin^2 \theta, \]
   \[ (1/r)dr/dt = -p(t) \cos^2 \theta + (1 - q(t)) \cos \theta \sin \theta. \]

   (b) Using part (a) discuss that if $q(t) > p^2(t)/4$ then solutions are oscillatory and if $q(t) < 0$ then solutions are nonoscillatory.