18.034 Solutions to Problemset 2

Spring 2009

1. \( y(x) = \begin{cases} -x + c_1 & x < 0 \\ x + c_2 & x > 0 \end{cases} \) for some \( c_1 \) and \( c_2 \).

\( y \) is continuous at \( x = 0 \) \( \Rightarrow \) \( c_1 = c_2 = 0 \).

But \( y(x) = \begin{cases} -x + c_1 & x < 0 \\ x + c_2 & x > 0 \end{cases} \) is not differentiable at \( x = 0 \).

2. As long as the solution is defined, \( \frac{y'}{y} = f(y) \) (y is never zero by uniqueness).

\[ \Rightarrow (\log |y|)' \leq |f(y)| \leq M \] by assumption. Then

\[ |y(x)| \leq |y_0|e^{Mx} \quad (1) \]

For \( a > 0 \), consider the rectangle \( \{(x, y) : |x| \leq a, |y| \leq |y_0|e^{Ma}\} \). If the solution does not exist on \( x \in (-a, a) \), then \( |y(a)| = |y_0|e^{Ma} \). This contradicts (2).

3. Let \( \alpha = a + ib, \beta = c + id \). In terms of polar coordinate functions,

\[ y'' = \text{Im}(\alpha \beta) \frac{\cos(\theta) \text{Re}((\beta + i\alpha)e^{i\theta})}{\text{Re}(\alpha e^{-i\theta})} \]

So \( y'' \) changes signs at slopes \(-b/a, \infty \) and \( \frac{b-c}{a+d} \).

4. (a) \( u \) solves the DE \( y' + (-b(x) + 2c(x)y_1(x))y + c(x)y^2 = 0 \).

(b) \( y_1(x) = x, u(x) = -\frac{1}{x+c} \).

5. (a) \( c_1 \sin x + c_2 \cos x \)

(b) \(- \sin 2x, 3, 2e^x \)

(c) \(-b \sin x - \sin 2x + 3 + 2e^x \)
6. (a) \( \ddot{u} + (p - 1)\dot{u} + qu = 0, \quad \cdot = \frac{d}{dt} \)

(b) \( \sin \log |x|, \cos \log |x| \)

(c) No solutions