UNIT I: FIRST-ORDER DIFFERENTIAL EQUATIONS

We set forth fundamental principles in the analysis of differential equations. We illustrate the use of integration to find the solutions of first-order linear differential equations and special classes of first-order nonlinear differential equations, called separable equations. Substitution techniques are used in studying linear fractional equations and special kind of second-order differential equations.

LECTURE 1. INTEGRATION AND SOLUTIONS

We recall the fundamental theorem of calculus

\[ \frac{d}{dx} \int_{x_0}^{x} f(s) \, ds = f(x), \]

if \( f \) is continuous on an interval \( x_0 \in I \). A solution of the differential equation

\[ \frac{dy}{dx} = f(x) \]

is the function \( y = \phi(x) \) which satisfies the differential equation on \( I \). Upon inspection of (1.1), then, \( y = \int_{x_0}^{x} f(s) \, ds \) is a solution of (1.2). This leads to an existence result.

**Theorem 1.1.** If \( f(x) \) is continuous on an interval \( x_0 \in I \) then given an arbitrary number \( y_0 \) there exists a unique solution of (1.2) satisfying \( y(x_0) = y_0 \). The solution is given as

\[ y(x) = y_0 + \int_{x_0}^{x} f(s) \, ds. \]

**Exercise.** Prove the uniqueness.

**Remark.** 1. The theorem specifies the interval of existence \( (x_0 \in I) \) and the class of functions considered (the class of continuous functions). It asserts the existence and uniqueness of a solution, prescribed the initial condition \( y(x_0) = y_0 \).

2. In the statement of the theorem, the interval of existence is \( I \), regardless of the initial condition. It is a special property of linear equations. For nonlinear equations, in general, the interval of existence depends on the initial value, e.g. the solution of the initial value problem

\[ \frac{dy}{dx} = y^2, \quad y(0) = y_0 \neq 0, \]

is given as \( y(x) = \frac{1}{(1/y_0) - x} \). It is defined on \( x \in [0, 1/y_0) \) for \( y_0 > 0 \).

3. The definite integral \( \int_{x_0}^{x} f(s) \, ds \) is defined as a limit of Riemann sums, as long as \( f \) is continuous; it doesn’t need to find a formal expression for the indefinite integral \( \int f(s) \, ds \) to give meaning to the definite integral, e.g. the error function \( \text{erf}(x) = \int_{0}^{x} e^{-s^2} \, ds \) and the sine integral function \( \text{Si}(x) = \int_{0}^{x} \sin(s)/s \, ds \) are commonly defined as definite integrals.
As an illustration, the solution of the initial value problem
\[ \frac{dy}{dx} = \sin^2 x, \quad y(0) = 0 \]
is given by the Fresnel sine integral function \( S(x) = \int_0^x \sin s^2 \, ds \). There is no elementary function \( F \) such that \( F'(x) = \sin x^2 \), but the function \( S(x) \) defined as a definite integral gives a perfectly good function.

The preceding discussion leads to how to solve differential equations of the form (1.2) by inspection. For any \( x_0 \), one solution is the function \( \int_{x_0}^x f(s) \, ds \). Other solutions are, then, obtained by adding an arbitrary constant \( C \) to this “particular” solution. Thus, the solutions of \( y' = e^{-x^2} \) are the functions \( y = \int e^{-s^2} \, ds = (\sqrt{\pi}/2)\text{erf}(x) + C \). From any one solution curve of (1.2), the others are obtained by the vertical translations \((x, y) \mapsto (x, y + C)\) and they form a one-parameter family of curves, one for each value of the parameter \( C \).

**Quadrature.** When the solution of a differential equation is expressed by a formula involving one or more integrations, it is said that the equation is solvable by quadrature. The term “quadrature” has its historical origin in the connection of integration with area. In plane geometry, a problem of quadrature, such as quadrature of the circle is a problem about the area of a plane figure. Not all differential equations can be solved by quadrature. In the following lecture, we will show that the first-order linear equation
\[ y' + p(x)y = q(x) \]
can be solved by quadrature. But, the second-order differential equation
\[ y'' + p(x)y' + q(x)y = r(x) \]
cannot be solved, in general, by quadrature, except for some special cases.

The next simple type of differential equation is
\[ (1.3) \quad \frac{dy}{dx} = g(y). \]
Such a differential equation is invariant under horizontal translations \((x, y) \mapsto (x + c, y)\). Geometrically, it means that any horizontal line is cut by all solution curves at the same angle (such lines are called “isoclines”). Therefore, if \( y = \phi(x) \) is a solution of (1.3), then so is \( y = \phi(x + c) \) for any \( c \). The differential equation (1.3) can be solved by writing it as \( dy/g(y) = dx \) and integrating.

**Example 1.2.** Consider
\[ (1.4) \quad \frac{dy}{dx} = y^2 - 1. \]
Since \( y^2 - 1 = (y - 1)(y + 1) \), the constant functions \( y = \pm 1 \) are particular solutions of (1.4). They are called steady states, stationary solutions or equilibria, in the sense that these solutions are independent of \( x \).

Next, if \( |y| < 1 \) then \( y^2 - 1 < 0 \), and follows \( dy/dx < 0 \). That is, the solution curve is decreasing. On the other hand, if \( |y| > 1 \), then \( dy/dx = y^2 - 1 > 0 \), and the solution curve is increasing. It gives us the qualitative behavior of solutions curves of (1.4).
Using the partial fractions and separating variables (we will discuss this technique in detail later), (1.4) is written as
\[ 2dx = dy \left( \frac{1}{y-1} - \frac{1}{y+1} \right). \]

Then, by integration, we obtain
\[ y(x) = \frac{1 \pm e^{2(x-c)}}{1 \mp e^{2(x-c)}} = \begin{cases} \tanh(c-x) \\ \coth(c-x) \end{cases} (c-x). \]

This procedure of separating variables “loses” the particular solutions \( y = \pm 1 \), but it gives all other solutions.

Note that if \( y = \phi(x) \) is a solution of (1.4) then so is \( 1/y = 1/\phi(x) \).

Exercise. Discuss \( y' = y^3 - y \).