LECTURE 26. EIGENVALUES AND EIGENVECTORS

We study the system

\[ y' = Ay, \]

where \( A = (a_{ij}) \) is a constant \( n \times n \) matrix.

When \( n = 1 \), the above system reduces to the scalar equation \( y' = ay \), and it has solutions of the form \( ce^{at} \). For \( n \geq 2 \), similarly, we try solutions of the form \( \bar{v} e^{\lambda t} \), where \( \bar{v} \in \mathbb{R}^n \) and \( \lambda \in \mathbb{C} \). Then, (26.1) becomes

\[ \lambda \bar{v} e^{\lambda t} = A \bar{v} e^{\lambda t}. \]

Subsequently,

\[ A \bar{v} = \lambda \bar{v}, \quad (A - \lambda I) \bar{v} = 0. \]

In order to find a solution (26.1) in the form of \( \bar{v} e^{\lambda t} \) we want to find a nonzero vector \( \bar{v} \) and \( \lambda \in \mathbb{C} \) satisfying (26.2). It leads to the following useful notions in linear algebra.

Definition 26.1. A nonzero vector \( \bar{v} \) satisfying (26.2) is called an eigenvector of \( A \) with the eigenvalue \( \lambda \in \mathbb{C} \).

These words are hybrids of English and German, and they follow German usage, “ei” rhymes with π.

We recognize that (26.2) is a linear system of equations for \( \bar{v} \). A well-known result from linear algebra is that it has a nontrivial solution \( \bar{v} \) if and only if \( A - \lambda I \) is singular. That is, \( p_A(\lambda) = \det(A - \lambda I) = 0 \), where \( p_A(\lambda) \) is the characteristic polynomial of \( A \). In this case, such a nontrivial solution \( \bar{v} \) is an eigenvector and the corresponding root of \( p_A(\lambda) = 0 \) is an eigenvalue.

Plane systems. For a \( 2 \times 2 \) matrix \( A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \), the characteristic polynomial is

\[ p(\lambda) = \det \begin{pmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{pmatrix} = \lambda^2 - (\text{tr} A) \lambda + \det A. \]

This quadratic polynomial has two roots, \( \lambda_1 \) and \( \lambda_2 \) (not necessarily distinct). Let \( \bar{v}_1 \) and \( \bar{v}_2 \) be the eigenvectors corresponding to the eigenvalues \( \lambda_1 \) and \( \lambda_2 \), respectively. By definition, \( \bar{v}_1 e^{\lambda_1 t} \) and \( \bar{v}_2 e^{\lambda_2 t} \) are solutions of (26.1).

If \( \lambda_1 \neq \lambda_2 \), then the functions \( \bar{v}_1 e^{\lambda_1 t} \) and \( \bar{v}_2 e^{\lambda_2 t} \) are linearly independent. Hence, they form a basis of solutions of (26.1). The general solution of (26.1) is given as

\[ \bar{y}(t) = c_1 \bar{v}_1 e^{\lambda_1 t} + c_2 \bar{v}_2 e^{\lambda_2 t}, \]

where \( c_1, c_2 \) are arbitrary constants. This shows one use of eigenvalues in the study of (26.1).

Let us define \( 2 \times 2 \) matrices

\[ V = (\bar{v}_1 \quad \bar{v}_2) \quad \text{and} \quad \Lambda = \text{diag} (\lambda_1, \lambda_2) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \]

One can verify that \( AV = V \Lambda \). If \( \bar{v}_1 \) and \( \bar{v}_2 \) are linearly independent, so that \( |V| \neq 0 \), then we can make the (non-singular) change of variables

\[ \bar{x} = V^{-1} \bar{y}. \]
Then, (26.1) is transformed into $\tilde{x}' = \Lambda \tilde{x}$, that is
\[ x'_1 = \lambda_1 x_1, \]
\[ x'_2 = \lambda_2 x_2. \]
That is, $\tilde{x}$ solves a decoupled system. The solution of this system is immediate and $x_1 = c_1 e^{\lambda_1 t}$, $x_2 = c_2 e^{\lambda_2 t}$. The new variables $\tilde{x}$ is called the canonical variables, and $\Lambda = V^{-1}AV$ is called the diagonalization. This is another use of eigenvalues. Canonical variables play a major role in engineering, economics, mechanics, and indeed in all fields that makes intensive use of linear systems with constant coefficients.

**Lemma 26.2.** If the eigenvalues of a $2 \times 2$ matrix are distinct, then the corresponding eigenvectors are linearly independent.

**Proof.** Suppose that the eigenvalues $\lambda_1 \neq \lambda_2$, but the eigenvectors satisfy
\[ c_1 \tilde{v}_1 + c_2 \tilde{v}_2 = 0. \]
We want to show that $c_1 = c_2 = 0$. Applying the matrix $A$ to (26.3), we obtain that
\[ c_1 \lambda_1 \tilde{v}_1 + c_2 \lambda_2 \tilde{v}_2 = 0. \]
Subtraction then yields
\[ c_1 (\lambda_2 - \lambda_1) \tilde{v}_1 = 0. \]
This implies $c_1 = 0$. Then, (26.3) implies $c_2 = 0$. \qed

**Example 26.3.** We consider $A = \begin{pmatrix} 12 & 5 \\ -6 & 1 \end{pmatrix}$.

Its characteristic polynomial is $p(\lambda) = \lambda^2 - 13\lambda + 42 = (\lambda - 6)(\lambda - 7)$, and $A$ has two distinct eigenvalues, $\lambda_1 = 6$ and $\lambda_2 = 7$.

If $\lambda_1 = 6$, then $A - 6I = \begin{pmatrix} 6 & 5 \\ -6 & -5 \end{pmatrix}$, and $\begin{pmatrix} 5 \\ -6 \end{pmatrix}$ is an eigenvector.

If $\lambda_2 = 7$, then $A - 7I = \begin{pmatrix} 5 & 5 \\ -6 & -6 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector. The general solution of (26.1) is, thus,
\[ y(t) = c_1 e^{6t} \begin{pmatrix} 5 \\ -6 \end{pmatrix} + c_2 e^{7t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \]
The canonical variable is $\tilde{x} = \begin{pmatrix} 5 \\ -6 \end{pmatrix}^{-1} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -y_1 - y_2 \\ 6y_1 + 5y_2 \end{pmatrix}$.

**Exercises.**
1. Show that $A = \begin{pmatrix} 6 & 1 \\ -1 & 8 \end{pmatrix}$ has only one eigenvalue $\lambda = 7$, and the only corresponding eigenvector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. In this case, we can’t construct the general solution of (26.1) from this.

2. Show that $A = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix}$ has two (complex) eigenvalues $1 \pm 2i$ and the corresponding eigenvectors $\begin{pmatrix} 1 \\ \pm 2i \end{pmatrix}$, respectively. They leads to the general (complex) solution of (26.1)
\[ c_1 e^{(1+2i)t} \begin{pmatrix} 1 \\ 2i \end{pmatrix} + c_2 e^{(1-2i)t} \begin{pmatrix} 1 \\ -2i \end{pmatrix}. \]
Higher-dimensional systems. If $A$ is an $n \times n$ matrix, where $n \geq 1$ is an integer, then

$$p_A(\lambda) = |A - \lambda I|$$

is a polynomial in $\lambda$ of degree $n$ and has $n$ roots, not necessarily distinct. That means, $A$ has $n$ eigenvalues $\lambda_1, \ldots, \lambda_n$, not necessarily distinct. Let $\vec{v}_1, \ldots, \vec{v}_n$ be the eigenvectors corresponding to $\lambda_1, \ldots, \lambda_n$, respectively. Let

$$V = (\vec{v}_1 \cdots \vec{v}_n) \quad \text{and} \quad \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n).$$

By definition, $\vec{v}_1 e^{\lambda_1 t}, \ldots, \vec{v}_n e^{\lambda_n t}$ are solutions of (26.1).

If $|V| \neq 0$, that means, if $\vec{v}_1, \ldots, \vec{v}_n$ are linearly independent, then they form a basis of solutions of (26.1). Moreover, $\vec{x} = V^{-1} \vec{y}$ is a canonical variable and $\vec{x}' = \Lambda \vec{x}$. In many cases, the vectors $\vec{v}_j$ can be chosen linearly independent even if $\lambda_j$ are not all distinct. Sometimes the condition $|V| \neq 0$ is met by the following.

**Lemma 26.4.** If eigenvalues $\lambda_1, \ldots, \lambda_n$ are distinct, then the corresponding eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$ are linearly independent.

**Proof.** Suppose not. Let $m > 1$ be the minimal number of vectors that are linearly dependent. Without loss of generality, we assume that $\vec{v}_1, \ldots, \vec{v}_m$ are linearly dependent, so that

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_m \vec{v}_m = 0$$

and some $c_j$ is nonzero. We further assume that $c_2 \neq 0$.

We now proceed similarly to Lemma 26.2. Applying $A$ to (26.5) we obtain

$$c_1 \lambda_1 \vec{v}_1 + c_2 \lambda_2 \vec{v}_2 + \cdots + c_m \lambda_m \vec{v}_m = 0.$$ 

Multiplying by $\lambda_1$ then

$$c_1 \lambda_1 \lambda_1 \vec{v}_1 + c_2 \lambda_2 \lambda_1 \vec{v}_2 + \cdots + c_m \lambda_m \lambda_1 \vec{v}_m = 0.$$ 

Thus, we have

$$c_2 (\lambda_2 - \lambda_1) \vec{v}_2 + \cdots + c_m (\lambda_m - \lambda_1) \vec{v}_m = 0.$$ 

Since $\vec{v}_2, \ldots, \vec{v}_m$ are linearly independent, $c_2 = \cdots = c_m = 0$ must hold. A contradiction then proves the assertion. \qed

We recall that if $A = A^T$ then the square matrix $A$ is called symmetric. If a complex matrix satisfies $A = A^*$, where $A^*$ denotes the conjugate transpose or adjoint of $A$, then $A$ is called Hermitian. A symmetric or a Hermitian matrix has many important properties pertaining to the study of (26.1) via eigenvalues.

(1) All eigenvalues of a symmetric matrix are real and eigenvalues of $A$ corresponding to different eigenvalues are orthogonal.

**Proof.** Let

$$A \vec{u} = \lambda \vec{u}, \quad A \vec{v} = \nu \vec{v},$$ 

$\vec{u}, \vec{v} \neq 0$ and $\lambda \neq \mu$. Then,

$$\vec{u}^T A \vec{u} = \lambda \vec{u}^T \vec{u}, \quad \vec{u}^T A \vec{v} = \bar{\lambda} \vec{u}^T \vec{v}.$$ 

Since $A = A^T$, it implies that $\lambda = \bar{\lambda}$. The second assertion is left as an exercise. \qed

(2) $A$ has $n$ linearly independent eigenvectors (regardless of the multiplicity of eigenvalues).

An immediate consequence of (1) and (2) is the following.

(3) If eigenvalues are simple (multiplicity = 1) then the corresponding eigenvectors are orthogonal.