18.034 Honors Differential Equations
Spring 2009

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1. Suppose $A$ is an $n \times n$ matrix and $y_1(t), y_2(t), \ldots y_n(t)$ are solutions to $y' = Ay$. Show that the set if $\{y_i(t_0)\}_{i=1}^n$ is linearly independent at some time $t_0$, then to any other solution $y(t)$ there correspond constants $c_i$ so that $y(t) = c_1y_1(t) + c_2y_2(t) + \ldots + c_ny_n(t)$ (i.e., the set $\{y_i(t)\}_{i=1}^n$ constitutes a basis of solutions).

2. Let $A$ be an $n \times n$ matrix.
   
   (a) Suppose $v_1$ and $v_2$ are eigenvectors of $A$ corresponding to the eigenvalues $\lambda_1$ and $\lambda_2$, respectively. If $\lambda_1 \neq \lambda_2$, show that $v_1$ and $v_2$ are linearly independent.
   
   (b) Assume now that $n = 2$. If $p_A(\lambda) = (\lambda - \lambda_1)^2$, show that either $A = \lambda_1 I$, or there is a unique eigenvector $v_1$ associated to $\lambda_1$ and a vector $v_2$ satisfying $(A - \lambda_1)v_2 = v_1$.
   
   (c) For $A$ as in the latter alternative in (2), show that the general solution to
   \[
   \frac{d}{dt}y = Ay
   \]
   is given by $y = e^{\lambda_1 t}(c_1 t + c_2)v_1 + c_1 e^{\lambda_1 t}v_2$.

3. For the system
   \[
   y'_1 = 3y_1 + 2y_2, \quad y'_2 = -2y_1 - y_2,
   \]
   find the unique fundamental matrix $U(t)$ satisfying $U(0) = I$.

4. Under what conditions on the trace and determinant of the $2 \times 2$ matrix $A$ will all solutions to the equation $y' = Ay$ satisfy $\lim_{t \to \infty} |y(t)| = 0$?