Part II Problems and Solutions

Problem 1: [Natural growth, separable equations] In recitation a population model was studied in which the natural growth rate of the population of oryx was a constant $k > 0$, so that for small time intervals $\Delta t$ the population change $x(t + \Delta t) - x(t)$ is well approximated by $kx(t)\Delta t$. (You also studied the effect of hunting them, but in this problem we will leave that aside.) Measure time in years and the population in kilo-oryx (ko).

A mysterious virus infects the oryxes of the Tana River area in Kenya, which causes the growth rate to decrease as time goes on according to the formula $k(t) = k_0/(a + t)^2$ for $t \geq 0$, where $a$ and $k_0$ are certain positive constants.

(a) What are the units of the constant $a$ in “$a + t$,” and of the constant $k_0$?
(b) Write down the differential equation modeling this situation.
(c) Write down the general solution to your differential equation. Don’t restrict yourself to the values of $t$ and of $x$ that are relevant to the oryx problem; take care of all values of these variables. Points to be careful about: use absolute values in $\int \frac{dx}{x} = \ln |x| + c$ correctly, and don’t forget about any “lost” solutions.
(d) Now suppose that at $t = 0$ there is a positive population $x_0$ of oryx. Does the progressive decline in growth rate cause the population stabilize for large time, or does it grow without bound? If it does stabilize, what is the limiting population as $t \to \infty$?

Solution: (a) The growth rate $k(t)$ has units years$^{-1}$ (so that $k(t)x(t)\Delta t$ has the same units as $x(t)\Delta t$). The variable $t$ has units years, so the $a$ added to it must have the same units, and $k_0$ must have units years in order for the units of the fraction to work out.

(b) $x(t + \Delta t) \approx x(t) + k(t)x(t)\Delta t$, so $\dot{x} = k_0x/(a + t)^2$.

(c) Separate: $dx/x = k_0/(a + t)^2 dt$. Integrate: $\ln |x| + c_1 = -k_0(a + t)^{-1} + c_2$. Amalgamate constants and exponentiate: $|x| = e^{c}e^{-k_0/(a+t)}$. Eliminate the absolute value: $x = Ce^{-k_0/(a+t)}$, where $C = \pm e^{c}$. Reintroduce the solution we lost by dividing by $x$ in the first step: allow $C = 0$. So the general solution is $x = Ce^{-k_0/(a+t)}$. (Note that the exponent $-k_0/(a + t)$ is dimensionless, as an exponent must be.)

(d) When $t$ gets very large, the exponent gets very near to zero, so there is a finite limiting population: $x_\infty = C$. Thus $x(t) = x_\infty e^{-k_0/(a+t)}$. Take $t = 0$ in the solution: $x_0 = x_\infty e^{-k_0/a}$, or $x_\infty = e^{k_0/a}x_0$. 
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