Part I Problems and Solutions

Problem 1: Change to polar form:

a) \(-1 + i\)

b) \(\sqrt{3} - i\)

Solution:

\[
a) \quad -1 + i = \sqrt{2}e^{i3\pi/4} \quad \text{b) } \sqrt{3} - i = 2e^{-i\pi/6}
\]

Problem 2: Express \(\frac{1 - i}{1 + i}\) in the form \(a + bi\) via two methods: one using the Cartesian form throughout, and one changing numerator and denominator to polar form. Show the two answers agree.

Solution: Using Cartesian form:

\[
\frac{1 - i}{1 + i} = \frac{(1 - i)^2}{(1 + i)(1 - i)} = \frac{-2i}{2} = -i
\]

Other way:

\[
\begin{align*}
1 - i &= \sqrt{2}e^{-i\pi/4} \\
1 + i &= \sqrt{2}e^{i\pi/4} \\
\frac{1 - i}{1 + i} &= \frac{\sqrt{2}}{\sqrt{2}} \cdot e^{(-\pi/4-\pi/4)i} \\
&= e^{-i\pi/2} = -i
\end{align*}
\]

Problem 3: Calculate each of the following two ways: first by using the binomial theorem and second by changing to polar form and using DeMoivre’s formula:

a) \((1 - i)^4\)

b) \((1 + i\sqrt{3})^3\)

Solution:

\[
a) \quad (1 - i)^4 = 1 + 4(-i) + 6(-i)^2 + 4(-i)^3 + (-i)^4 \\
= 1 - 6 + 1 + i(-4 + 4) = -4
\]
By DeMoivre:

$$1 - i = \sqrt{2}e^{-i\pi/4}$$

$$(1 - i)^4 = (\sqrt{2})^4e^{-i\pi} = 4 \cdot (-1) = -4$$

$$b) \quad (1 + i\sqrt{3})^3 = 1 + 3(i\sqrt{3}) + 3(i\sqrt{3})^2 + (i\sqrt{3})^3$$
$$= 1 + 3i\sqrt{3} + 3 \cdot (-3) + i^3\cdot\sqrt{3}$$
$$= -8 + i(3\sqrt{3} - 3\sqrt{3}) = -8$$

By DeMoivre:

$$1 + i\sqrt{3} = 2e^{i\pi/3}$$

$$(1 + i\sqrt{3})^3 = 8e^{i\pi} = -8$$

**Problem 4:** Express the 6 sixth roots of 1 in the form $a + bi$.

**Solution:** In polar form the sixth roots of 1 are $e^{i\pi k/3}$ where $k = 0, 1, 2, \ldots, 5$. Thus (using $\cos(\pi/3) = 1/2, \sin(\pi/3) = \sqrt{3}/2$) the roots in Cartesian form are

$$\pm 1, \quad \text{and} \quad \pm 1 \pm i\sqrt{3} \quad \text{over} \quad 2$$

**Problem 5:** Solve the equation $x^4 + 16 = 0$

**Solution:**

$$\sqrt[4]{-16} = 2 \cdot \sqrt{-1}$$

The 4th roots of $-1$ are $e^{i(\pi/4+n\pi/2)} = \pm 1 \pm i$. They are shown in the figure at right:

Thus, $\sqrt{2}(\pm 1 \pm i)$ are the roots of $x^4 + 16 = 0$. 

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