Finding $n$-th Roots

To solve linear differential equations with constant coefficients, we need to be able to find the real and complex roots of polynomial equations. Though a lot of this is done today with calculators and computers, one still has to know how to do an important special case by hand: finding the roots of

$$z^n = \alpha,$$

where $\alpha$ is a complex number, i.e., finding the $n$-th roots of $\alpha$. Polar representation will be a big help in this.

Let’s begin with a special case: the $n$-th roots of unity: the solutions to

$$z^n = 1.$$

To solve this equation, we use polar representation for both sides, setting $z = re^{i\theta}$ on the left, and using all possible polar angles on the right; using the exponential law to multiply, the above equation then becomes

$$r^n e^{i n \theta} = 1 \cdot e^{i(2k\pi)}, \quad k = 0, \pm 1, \pm 2, \cdots.$$

Equating the absolute values and the polar angles of the two sides gives

$$r^n = 1, \quad n\theta = 2k\pi, \quad k = 0, \pm 1, \pm 2, \cdots,$$

from which we conclude that

$$r = 1, \quad \theta = \frac{2k\pi}{n}, \quad k = 0, 1, \cdots, n - 1. \quad (1)$$

In the above, we get only the value $r = 1$, since $r$ must be real and non-negative. We don’t need any integer values of $k$ other than $0, \cdots, n - 1$, since they would not produce a complex number different from the above $n$ numbers. That is, if we add $an$, an integer multiple of $n$, to $k$, we get the same complex number:

$$\theta' = \frac{2(k + an)\pi}{n} = \theta + 2a\pi, \quad \text{and} \quad e^{i\theta'} = e^{i\theta}, \quad \text{since} \ e^{2a\pi i} = (e^{2\pi i})^a = 1.$$

We conclude from (1) therefore that

the $n$-th roots of $1$ are the numbers $e^{2k\pi i/n}$, $k = 0, \cdots, n - 1$. \quad (2)
This shows there are \( n \) complex \( n \)-th roots of unity. They all lie on the unit circle in the complex plane, since they have absolute value 1; they are evenly spaced around the unit circle, starting with the root \( z = 1 \); the angle between two consecutive roots is \( 2\pi/n \). These facts are illustrated for the case \( n = 6 \) in the figure below.

Fig. 1. The six solutions to the equation \( z^6 = 1 \) lie on a unit circle in the complex plane.

From (2), we get another notation for the roots of unity (\( \zeta \) is the Greek letter “zeta”):

\[
\text{the } n\text{-th roots of } 1 \text{ are } 1, \zeta, \zeta^2, \ldots, \zeta^{n-1}, \text{ where } \zeta = e^{2\pi i/n}. \tag{3}
\]

We now generalize the above to find the \( n \)-th roots of an arbitrary complex number \( w \). We begin by writing \( w \) in polar form:

\[
w = re^{i\theta}; \quad \theta = \text{Arg}w, \quad 0 \leq \theta < 2\pi,
\]
i.e., \( \theta \) is the principal value of the polar angle of \( w \). Then the same reasoning as we used above shows that if \( z \) is an \( n \)-th root of \( w \), then

\[
z^n = w = re^{i\theta} \quad \text{so} \quad z = r^{1/n}e^{i(\theta+2k\pi)/n}, \quad k = 0, 1, \ldots, n-1. \tag{4}
\]

Comparing this with (3), we see that these \( n \) roots can be written in the suggestive form

\[
\sqrt[n]{w} = z_0, z_0\zeta, z_0\zeta^2, \ldots, z_0\zeta^{n-1}, \quad \text{where } z_0 = \sqrt[n]{r}e^{i\theta/n}. \tag{5}
\]

As a check, we see that all of the \( n \) complex numbers in (5) satisfy \( z^n = w \):
Finding $n$-th Roots

$$\left(z_0^2\right)^n = z_0^n \cdot \frac{\pi i}{n} = z_0^n \cdot 1^i,$$

since $\zeta^n = 1$, by (3); by the definition (5) of $z_0$ and (4).

Example. Find in Cartesian form all values of

a) $\sqrt[3]{1}$  

Solution. a) According to (3), the cube roots of 1 are $1, \omega, \text{ and } \omega^2$, where

$$\omega = e^{2\pi i/3} = \cos(2\pi/3) + i \sin(2\pi/3) = -\frac{1}{2} + \frac{i\sqrt{3}}{2},$$

$$\omega^2 = e^{-2\pi i/3} = \cos(-2\pi/3) + i \sin(-2\pi/3) = -\frac{1}{2} - \frac{i\sqrt{3}}{2}.$$

The greek letter $\omega$ (“omega”) is traditionally used for this cube root. Note that for the polar angle of $\omega^2$ we used $-2\pi/3$ rather than the equivalent angle $4\pi/3$, in order to take advantage of the identities

$$\cos(-x) = \cos(x) \quad \text{and} \quad \sin(-x) = -\sin(x).$$

Note that $\omega^2 = \overline{\omega}$. Another way to do this problem would be to draw the position of $\omega^2$ and $\omega$ on the unit circle and use geometry to figure out their coordinates.

b) To find $\sqrt[3]{i}$, we can use (5). We know that $\sqrt[3]{1} = 1, i, -1, -i$ (either by drawing the unit circle picture or by using (3)). Therefore by (5), we get

$$\sqrt[3]{i} = z_0, z_0i, -z_0, -z_0i,$$

where $z_0 = e^{\pi i / 8} = \cos(\pi/8) + i \sin(\pi/8)$;

$$= a + ib, -b + ia, -a - ib, b - ia \quad \text{where } z_0 = a + ib = \cos(\pi/8) + i \sin(\pi/8).$$

Example. Solve the equation $x^6 - 2x^3 + 2 = 0$.

Solution. Treating this as a quadratic equation in $x^3$, we solve the quadratic by using the quadratic formula; the two roots are $1 + i$ and $1 - i$ (check this!), so the roots of the original equation satisfy either

$$x^3 = 1 + i \quad \text{or} \quad x^3 = 1 - i.$$

This reduces the problem to finding the cube roots of the two complex numbers $1 \pm i$. We begin by writing them in polar form:

$$1 + i = \sqrt{2}e^{\pi i / 4}, \quad 1 - i = \sqrt{2}e^{-\pi i / 4}.$$

(Once again, note the use of the negative polar angle for $1 - i$, which is more convenient for calculations.) The three cube roots of the first of these are (by (4)),
\[ \sqrt[3]{2}e^{\pi i/12} = \sqrt[3]{2}(\cos(\pi/12) + i\sin(\pi/12)) \]
\[ \sqrt[3]{2}e^{3\pi i/4} = \sqrt[3]{2}(\cos(3\pi/4) + i\sin(3\pi/4)), \quad \text{since} \quad \frac{\pi}{12} + \frac{2\pi}{3} = \frac{3\pi}{4}; \]
\[ \sqrt[3]{2}e^{-7\pi i/12} = \sqrt[3]{2}(\cos(7\pi/12) - i\sin(7\pi/12)), \quad \text{since} \quad \frac{\pi}{12} - \frac{2\pi}{3} = -\frac{7\pi}{12}. \]

The second cube root can also be written as \[ \sqrt[3]{2} \left( \frac{-1 + i}{\sqrt{2}} \right) = -\frac{1 + i}{\sqrt{2}}. \]

This gives three of the cube roots. The other three are the cube roots of \( 1 - i \), which may be found by replacing \( i \) by \( -i \) everywhere (i.e., taking the complex conjugate).

The cube roots can also be described according to (5) as \( z_1, z_1\omega, z_1\omega^2 \) and \( z_2, z_2\omega, z_2\omega^2 \) where \( z_1 = \sqrt[3]{2}e^{\pi i/12}, z_2 = \sqrt[3]{2}e^{-\pi i/12}. \)