Part II Problems and Solutions

Problem 1:  [Sinusoidal input and output]

(a) Express \( \text{Re} \left( \frac{e^{3it}}{\sqrt{3} + i} \right) \) in the form \( a \cos(3t) + b \sin(3t) \). Then rewrite this in the form \( A \cos(3t - \phi) \). Now find this same answer using the following method. By finding its modulus and argument, write \( \sqrt{3} + i \) in the form \( A e^{i\phi} \). Then substitute this into \( e^{3it} / (\sqrt{3} + i) \), and use properties of the exponential function to find \( B \) and \( \phi \) such that \( \frac{e^{3it}}{\sqrt{3} + i} = Be^{i(3t - \phi)} \). Finally, take the real part of this new expression.

(b) Find a solution to the differential equation \( \dot{z} + 3z = e^{2it} \) of the form \( we^{2it} \), where \( w \) is some complex number. What is the general solution?

(c) Find a solution of \( \dot{x} + 3x = \cos(2t) \) by relating this ODE to the one in (b). What is the general solution?

Solution:  (a) \( \frac{e^{3it}}{\sqrt{3} + i} = \frac{(\sqrt{3} - i)}{4} (\cos(3t) + i \sin(3t)) \) has real part \( \frac{\sqrt{3}}{4} \cos(3t) + \frac{1}{4} \sin(3t) \).

Form the right triangle with sides \( a = \sqrt{3} \) and \( b = \frac{1}{4} \). The hypotenuse is \( A = 1/2 \) and the angle is \( \phi = \pi / 6 \).

\( \sqrt{3} + i = 2e^{\pi i/6} \) (by essentially the same triangle), so \( \frac{e^{3it}}{\sqrt{3} + i} = \frac{1}{2} e^{i(3t - \pi/6)}; B = \frac{1}{2}, \phi = \frac{\pi}{6} \), and \( \text{Re}(Be^{i(3t - \phi)}) = B \cos(3t - \phi) \), so you get the same answer.

(b) Substituting \( z = we^{2it} \), \( e^{2it} = w2ie^{2it} + 3we^{2it} \), so \( 1 = w(2i + 3) \) or \( w = \frac{1}{2i+3} \). Thus a solution of the desired form is \( z_p = \frac{1}{2i+3} e^{2it} \). The general solution is \( z = z_p + ce^{-3t} \).

(c) If \( x = \text{Re}(z) \), the real part of \( \dot{z} + 3z = e^{2it} \) is \( \dot{x} + 3x = \cos(2t) \). So we are looking for \( \text{Re}(z_p) \), where \( z_p \) is the answer in part (b).

In polar form, \( 2i + 3 = \sqrt{13}e^{i\phi} \), where \( \phi = \tan^{-1}(2/3) \). Thus,

\( z_p = \frac{1}{\sqrt{13}} e^{i(2t - \phi)} \)

We get

\( x_p = \text{Re}(z_p) = \frac{1}{\sqrt{13}} \cos(2t - \phi) \).

The general solution is then \( x = x_p + ce^{-3t} \).