

## Errors In Euler's Method

As we have seen with the applet, Euler's method is rarely exact. In this section we try to understand potential sources of error, and find ways to estimate or bound it.

### 1. Common Error Sources

Let us stress this again: the Euler polygon is not an exact solution; the direction field at its vertices usually differs more and more from the direction field along the actual solution. At places where the direction field is changing rapidly, this can quickly produce very bad approximations: the variation of the direction field causes the integral curve to bend away from its approximating Euler strut. One trick ODE solvers use is to take smaller step sizes when the direction field is steep.

As a general rule Euler's method becomes more accurate the smaller the step-size  $h$  is taken. However, if  $h$  is too small, round-off errors can appear and will accumulate, particularly on a pocket calculator; whenever in doubt, try to do all computations on a computer, keeping a high number of significant figures.

### 2. Estimating the Sign of the Error: Concave and Convex Functions

Can we predict whether the Euler approximation is too big or too small? In our first example when exploring the Euler Method applet they were too small. Pictorially, this was because the solution was curving upwards, leaving the polygons below it. How would we know without a picture whether a solution is "curving up" or "curving down"?

The mathematical concept corresponding to *curving up* is convexity: a function  $y(x)$  is called *convex* on an interval if  $y''(x) > 0$  on that interval. *Curving down* is called concavity; the corresponding condition is  $y''(x) < 0$ . Any tangent to a convex function at a point lies below the function at that point. Thus, intuitively, if  $y''(x_0) > 0$ , Euler's estimate at  $x_1$  is likely to be too low. Similarly, if  $y''(x_0) < 0$ , it is likely to be too high.

Let us do a worked example.

**Example.** We'll use Euler's method to estimate the value at  $x = 1.5$  of the solution to  $y' = f(x, y) = y^2 - x^2$  with  $y(0) = -1$ , using  $h = 0.5$ . We get the table:

$n$	$x_n$	$y_n$	$f(x_n, y_n)$	$hf(x_n, y_n)$
0	0	-1	1	0.5
1	0.5	-0.5	0	0
2	1.0	-0.5	-0.75	-0.375
3	1.5	-0.875		

We want to use the intuition developed in the paragraph above; how can we compute  $y''$ ? Differentiate both sides of the equation  $y' = y^2 - x^2$  with respect to  $x$  and use the chain rule for the term  $\frac{dy^2}{dx}$ . to get:

$$y'' = 2yy' - 2x$$

Thus  $y''(0) = 2(-1)(1) - 0 = -2$ . This means that the estimate is likely to be too large.

First:  $y = x - 1$  with slope 1, so  $y' = y^2 - x^2 = (x - 1)^2 - x^2 = -2x + 1$ . When  $x$  is in the interval  $[0, 0.5]$  we have  $-2x + 1 \leq 1$ .

Second:  $y = -0.5$  with slope zero, so  $y' = y^2 - x^2 = 0.25 - x^2$ . When  $x$  is in  $[0.5, 1]$ , this is nonpositive.

Third:  $y = -0.75x + 0.25$  with slope  $-0.75$ , so  $y' = y^2 - x^2 = -0.4375x^2 - 0.375x + 0.0625$ . We would like to compare this with  $-0.75$  in the interval  $[1, 1.5]$ . At  $x = 1$ , we have equality, so it suffices to show that the first derivative of  $y'$  on the segment, or  $y'' = -0.875x - 0.375$ , is nonpositive for  $x$  in  $[1, 1.5]$ , and it is.

### 3. Two cautionary tales

#### 3.1. Dramatic overshoot

In the third example in our Euler Applet exploration we looked at the DE  $y' = y^2 - x$  starting at  $(-0.98, 0)$ . The actual solution was drastically different from any of the Euler polygons. This phenomenon is best explained in the language graphical solutions. There is a separatrix for  $y' = y^2 - x$  that passes just above  $(-0.98, 0)$ ; you can test this by playing around with the applet a little. The integral curve through  $(-0.98, 0)$  remains below the separatrix; however, all the Euler polygons cross it, and subsequently behave like integral curves from the other side of the separatrix.

Depending on the initial point, this sort of phenomenon can be very difficult to avoid; one strategy can be to first study the equation using geometric techniques.

### 3.2. Divergent estimates

Consider the IVP  $y' = y^2$ ,  $y(0) = 1$ . Let us try to estimate  $y(1)$  using Euler's method. For  $h = 0.2$ , we get

$n$	$x_n$	$y_n$
0	0	1
1	0.2	1.2
2	0.4	1.49
3	0.6	1.93
4	0.8	2.68
5	1.0	4.11

(We omit the columns with  $f(x_n, y_n)$  and  $f(x_n, y_n)h$ .)

For smaller step sizes, we get the following estimates:

$h$	Estimate for $y(1)$
0.1	37.6
0.05	91.25
0.02	238.21

What is going on? We can actually solve this equation explicitly, for instance with the separation of variables method of session one. The solution is:

$$y(x) = 1/(1 - x).$$

This is not defined for  $x = 1$ : as  $x \rightarrow 1^-$ ,  $y \rightarrow +\infty$ .

The lesson is that in practice, one should never simply choose a step size and accept the answer. You should try smaller and smaller  $h$  until the answer settles down. If it does, you have one good bit of evidence to accept the approximation; if it doesn't, the method has failed. The computer does not eliminate the need to think!

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