The Sinusoidal Identity

The sum of two sinusoidal functions of the same frequency is another sinusoidal function with that frequency! For any real constants $a$ and $b$,

$$a \cos(\omega t) + b \sin(\omega t) = A \cos(\omega t - \phi) \quad (1)$$

where $A$ and $\phi$ can be described in at least two ways:

$$A = \sqrt{a^2 + b^2}, \quad \phi = \tan^{-1} \frac{b}{a}; \quad (2)$$

$$a + bi = Ae^{i\phi}. \quad (3)$$

Conversely, we have

$$a = A \cos(\phi) \quad \text{and} \quad b = A \sin(\phi). \quad (4)$$

Geometrically this is summarized by the triangle in the figure below.

![Fig. 1. $a + bi = Ae^{i\phi}$.](image)

One proof of (1) is a simple application of the cosine addition formula

$$\cos(\alpha - \beta) = \cos(\alpha) \cos(\beta) + \sin(\alpha) \sin(\beta).$$

We will now give an equivalent proof using Euler’s formula and complex arithmetic: The triangle in Figure 1 is the standard polar coordinates triangle. It shows $a + ib = Ae^{i\phi}$ or $a - ib = Ae^{-i\phi}$. Thus

$$A \cos(\omega t - \phi) = \text{Re}(Ae^{i(\omega t - \phi)})$$

$$= \text{Re}(e^{i\omega t} \cdot Ae^{-i\phi})$$

$$= \text{Re}((\cos(\omega t) + i \sin(\omega t)) \cdot (a - ib))$$

$$= \text{Re}(a \cos(\omega t) + b \sin(\omega t) + i(a \sin(\omega t) - b \cos(\omega t)))$$

$$= a \cos(\omega t) + b \sin(\omega t).$$

We should stress the importance of the trigonometric identity (1). It shows that any linear combination of $\cos(\omega t)$ and $\sin(\omega t)$ is not only periodic of
period $\frac{2\pi}{\omega}$, but is also sinusoidal. If you try to add $\cos(\omega t)$ to $\sin(\omega t)$ “by hand”, you will probably agree that this is not at all obvious.

We will call $A \cos(\omega t - \phi)$ amplitude-phase form and $a \cos(\omega t) + b \sin(\omega t)$ rectangular or Cartesian form. You should be familiar with amplitude-phase form; we usually prefer it because both amplitude and phase have geometric and physical meaning for us.