Part I Problems and Solutions

In each of the following three problems find a particular solution to the differential equation. Use complex exponentials where possible.

Problem 1: \( y^{(3)} + y'' - y' + 2y = 2 \cos x \)

Solution: characteristic polynomial \( p(s) = s^3 + s^2 - s + 2; \) complex replacement: \( z^{(3)} + z'' - 2z' + 2z = 2e^{ix} \). Complex solution from ERF:

\[
z_p = \frac{2e^{ix}}{p(i)} = \frac{2e^{ix}}{i^3 + i^2 - i + 2} = \frac{2(1 + 2i)e^{ix}}{(1 - 2i)(1 + 2i)}
\]

Thus

\[
z_p = \frac{2 + 4i}{5} (\cos x + i \sin x)
\]

, and

\[
y_p = \text{Re}(z_p) = \frac{2}{5} \cos x - \frac{4}{5} \sin x
\]

Problem 2: \( y'' - 2y' + 4y = e^x \cos x \)

Solution: complex replacement: \( z'' - 2z' + 4z = e^{(1+i)x} \). \( p(s) = s^2 - 2s + 4; \) \( p(1 + i) = (1 + i)^2 - 2(1 + i) + 4 = 2 \), so \( z_p = \frac{e^{(1+i)x}}{2} \), and thus

\[
y_p = \text{Re}(z_p) = \frac{1}{2} e^x \cos x
\]

Problem 3: \( y'' - 6y' + 9y = e^{3x} \)

Solution: \( p(s) = s^2 - 6s + 9 = (s - 3)^2 \) so \( y_p = cx^2 e^{3x} \). \( p(3) = p'(3) = 0, p''(3) = 2 \neq 0. \) This the generalized ERF gives

\[
y_p = \frac{1}{2} x^2 e^{3x}
\]

Problem 4: Find the real general solution to the DE

\[
\frac{d^3x}{dt^3} - x = e^{2t}
\]
Solution: \( p(r) = r^3 - 1, \quad p(2) = 2^3 - 1 = 7 \neq 0 \). Thus the ERF gives a particular solution to the inhomogenous DE \( x^{(3)} - x = e^{2t} \) as \( x_p = \frac{1}{p(2)} e^{2t} = \frac{1}{7} e^{2t} \).

\( x_h \) solution to the homogeneous eqn. \( x^{(3)} - x = 0 \), characteristic equation \( p(r) = r^3 - 1 = 0 \to r^3 = 1 \to \) cube roots of 1.

\( r = 1, e^{2\pi i/3}, e^{4\pi i/3} = 1, \ -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \ -\frac{1}{2} - i\frac{\sqrt{3}}{2} \)

So \( x_1 = e^t, \quad x_2 = e^{-t/2} \cos \left( \frac{\sqrt{3}}{2} t \right), \quad x_3 = e^{-t/2} \sin \left( \frac{\sqrt{3}}{2} t \right) \)

Real solutions: \( x_h = c_1 x_1 + c_2 x_2 + c_3 x_3 \) so the general solution \( x = x_p + x_h \) is

\[
    x = \frac{1}{7} e^{2t} + c_1 e^t + c_2 e^{-t/2} \cos \left( \frac{\sqrt{3}}{2} t \right) + c_3 e^{-t/2} \sin \left( \frac{\sqrt{3}}{2} t \right)
\]

Problem 5: Find a particular solution to the differential equation

\[
y'' - 4y = \frac{1}{2} (e^{2x} + e^{-2x})
\]

Solution: \( y_p = y_{p,1} + y_{p,2} \) where \( p(D)y_{p,1} = \frac{1}{2} e^{2x} \) and \( p(D)y_{p,2} = \frac{1}{2} e^{-2x} \) by superposition principle. \( p(r) = r^2 - 4 \) so \( p(2) = p(-2) = 0 \). Using the generalized ERF \( p'(r) = 2r \). Thus,

\[
y_{p,1} = \frac{1}{p'(2)} x e^{2x} = \frac{1}{8} x e^{2x}
\]

\[
y_{p,2} = \frac{1}{p'(-2)} x e^{-2x} = -\frac{1}{8} x e^{-2x}
\]

\[
y_p = y_{p,1} + y_{p,2} = \frac{x}{8} (e^{2x} - e^{-2x})
\]