Generalized Exponential Response Formula

We can also solve an LTI DE $p(D)x = q(t)$ with exponential input $q(t) = Be^{at}$ even when $p(a) = 0$. The answer is given by the following generalized Exponential Response formula (the proof of which we postpone to the session on Linear Operators.

**Generalized Exponential Response Formula.** Let $p(D)$ be a polynomial operator with constant coefficients, and $p^{(s)}$ its $s$-th derivative. Then

$$p(D)x = Be^{at}, \quad \text{where } a \text{ is real or complex}$$

has the particular solution

$$x_p = \begin{cases} 
\frac{Be^{at}}{p(a)} & \text{if } p(a) \neq 0 \\
\frac{Bte^{at}}{p'(a)} & \text{if } p(a) = 0 \text{ and } p'(a) \neq 0 \\
\frac{Bt^2e^{at}}{p''(a)} & \text{if } p(a) = p'(a) = 0 \text{ and } p''(a) \neq 0 \\
\cdots \\
\frac{Bt^se^{at}}{p^{(s)}(a)} & \text{if } a \text{ is an } s\text{-fold zero}
\end{cases}$$

**Note:** Later when we cover resonance the case $p(a) = 0, p'(a) \neq 0$ will be called the Resonant Response Formula

**Example 1.** Find a particular solution to the equation

$$\ddot{x} + 8\dot{x} + 15x = e^{-5t}$$

**Solution.** The characteristic polynomial is $p(r) = r^2 + 8r + 15$. Since $p(-5) = 0$ we need to use the generalized ERF. Computing $p'(r) = 2r + 8$, which implies $p'(-5) = -2$. Therefore the generalized ERF gives

$$x_p = \frac{te^{-5t}}{p'(-5)} = -\frac{te^{-5t}}{2}.$$ 

**Example 2.** Find a particular solution to

$$\ddot{x} + 2\dot{x} + 2x = e^{-t} \cos t.$$
**Solution.** First we complexify the equation

\[ \ddot{z} + 2\dot{z} + 2z = e^{(-1+i)t}, \quad \text{where} \quad x = \text{Re}(z). \]

The characteristic polynomial is \( p(r) = r^2 + 2r + 2 \). Computing,

\[ p(-1+i) = (-1+i)^2 + 2(-1+i) + 2 = 0, \quad p'(r) = 2r + 2, \quad p'(-1+i) = 2i. \]

Since \( p(-1+i) = 0 \) we use the generalized ERF

\[ z_p = \frac{te^{(-1+i)t}}{p'(-1+i)} = \frac{te^{(-1+i)t}}{2i} = \frac{te^{-(\cos t + i \sin t)}}{2i} \]

Finally we take the real part to get

\[ x_p = \text{Re}(z_p) = \frac{te^{-t} \sin t}{2}. \]