Proof of the Generalized Exponential Response Formula

Using the exponential shift rule, we can now give a proof of the general case of the ERF which we stated without proof in the session on Exponential Response. This is a slightly complicated proof and you can safely skip it if you are not interested.

**Generalized Exponential Response Formula.** Let \( p(D) \) be a polynomial operator with constant coefficients and \( p^{(s)} \) its s-th derivative. Then

\[
p(D)x = e^{at}, \quad \text{where } a \text{ is real or complex} \tag{1}
\]

has the particular solution

\[
x_p = \begin{cases} 
  i) & \frac{e^{at}}{p(a)} \quad \text{if } p(a) \neq 0 \\
  ii) & \frac{te^{at}}{p'(a)} \quad \text{if } p(a) = 0 \text{ and } p'(a) \neq 0 \\
  iii) & \frac{t^2e^{at}}{p''(a)} \quad \text{if } p(a) = p'(a) = 0 \text{ and } p''(a) \neq 0 \\
  \cdots \\
  iv) & \frac{t^se^{at}}{p^{(s)}(a)} \quad \text{if } a \text{ is an s-fold zero}
\end{cases}
\]

**Proof.** That (i) is a particular solution to (1) follows immediately by using the linearity and substitution rules given earlier.

\[
p(D)x_p = p(D)\frac{e^{at}}{p(a)} = \frac{1}{p(a)} p(D)e^{at} = \frac{p(a)e^{at}}{p(a)} = e^{at}.
\]

Since cases (ii) and (iii) are special cases of (iv) we skip right to that. For case (iv), we begin by noting that to say the polynomial \( p(D) \) has the number \( a \) as an s-fold zero is the same as saying \( p(D) \) has a factorization

\[
p(D) = q(D)(D - a)^s, \quad q(a) \neq 0. \tag{2}
\]

We will first prove that (2) implies

\[
p^{(s)}(a) = q(a)s!. \tag{3}
\]
To prove this, let $k$ be the degree of $q(D)$ and write it in powers of $(D - a)$:

\[ q(D) = q(a) + c_1(D - a) + \ldots + c_k(D - a)^k; \quad \text{then} \]
\[ p(D) = q(a)(D - a)^s + c_1(D - a)^{s+1} + \ldots + c_k(D - a)^{s+k}; \quad (4) \]
\[ p^{(s)}(D) = q(a) s! + \text{positive powers of } D - a. \]

Substituting $a$ for $D$ on both sides proves (3). □

Using (3), we can now prove (iv) easily using the exponential-shift rule. We have

\[
p(D) \frac{e^{at}x^s}{p^{(s)}(a)} = \frac{e^{at}}{p^{(s)}(a)} p(D + a) x^s, \quad \text{by linearity and ERF case (i)};
\]
\[
= \frac{e^{at}}{p^{(s)}(a)} q(D + a) D^s x^s, \quad \text{by (2)};
\]
\[
= \frac{q(a) s!}{q(a) s!} q(D + a) s!, \quad \text{by (3)};
\]
\[
= \frac{e^{at}}{q(a) s!} q(a) s! = e^{at},
\]

where the last line follows from (4), since $s!$ is a constant:

\[ q(D + a)s! = (q(a) + c_1 D + \ldots + c_k D^k) s! = q(a)s!. \]

Note: By linearity we could have stated the formula with a factor of $B$ in the input and a corresponding factor of $B$ to the output. That is, the DE

\[ p(D)x = Be^{at} \]

has a particular solution

\[ x_p = \frac{Be^{at}}{p(a)}, \quad \text{if } p(a) \neq 0 \text{ etc.} \]