1. Check that both \( x = \cos(\omega t) \) and \( x = \sin(\omega t) \) satisfy the second order linear differential equation
\[
\ddot{x} + \omega^2 x = 0
\]
This equation is called the harmonic oscillator.

If \( x = \cos(\omega t) \), then \( \dot{x} = -\omega \sin(\omega t) \) and \( \ddot{x} = -\omega^2 \cos(\omega t) = -\omega^2 x \). If \( x = \sin(\omega t) \), then \( \dot{x} = \omega \cos(\omega t) \) and \( \ddot{x} = -\omega^2 \sin(\omega t) = -\omega^2 x \).

2. In fact, check that any sinusoidal function with circular frequency \( \omega \), \( A \cos(\omega t - \phi) \), satisfies the equation \( \ddot{x} + \omega^2 x = 0 \).

If \( x = A \cos(\omega t - \phi) \), then \( \dot{x} = -A \omega \sin(\omega t - \phi) \), and \( \ddot{x} = -A \omega^2 \cos(\omega t - \phi) = -\omega^2 x \).

3. Among the functions \( x(t) = A \cos(\omega t - \phi) \), which have \( x(0) = 0 \)? Doesn’t this contradict the uniqueness theorem for differential equations?

\( x(0) = A \cos \phi \). When \( A = 0 \), then \( x(t) = 0 \) for every \( t \); when \( A \neq 0 \), \( x(0) = 0 \) implies \( \cos \phi = 0 \), and hence \( \phi \) can be any odd multiple of \( \pi/2 \). So, up to sign, the solutions that satisfy the given initial condition are \( x(t) = A \cos(\omega t - \pi/2) = A \sin(\omega t) \), where \( A \neq 0 \) can be arbitrary.

This does not contradict the uniqueness theorem, because the uniqueness theorem as stated only applies to first order equations.

4. Given numbers \( x_0 \) and \( \dot{x}_0 \), can you find a solution to \( \ddot{x} + \omega^2 x = 0 \) for which \( x(0) = x_0 \) and \( x(0) = \dot{x}_0 \)? How many such solutions are there?

The general solution to this differential equation is \( x(t) = a \cos(\omega t) + b \sin(\omega t) \).

Taking into account the given initial conditions, have \( x(0) = a \), so \( a = x_0 \), and \( x'(0) = -a \omega \sin 0 + b \omega \cos 0 = b \omega = x_0 \), so \( b = x_0/\omega \). That is, the solution that satisfies the initial conditions is \( x = x_0 \cos(\omega t) + \frac{x_0}{\omega} \sin(\omega t) \), and there is only one such solution.

5. Let \( r \) denote a constant, which is perhaps complex valued. Suppose that \( e^{rt} \) is a solution to \( \ddot{x} + kx = 0 \). What does \( r \) have to be?

Let \( x = e^{rt} \). Then \( \dot{x} = re^{rt} \), \( \ddot{x} = r^2 e^{rt} \), and \( \ddot{x} + kx = (r^2 + k)e^{rt} \). We want this to be zero. Since \( e^{rt} \) is never zero, the other factor must be zero, and \( r^2 = -k \). That is, \( r \) must have the form \( \pm i \sqrt{k} \) (this will be two real numbers if \( k < 0 \)).

6. Find a solution \( x_1 \) to \( \ddot{x} - a^2 x = 0 \) [note the sign!] such that \( x_1(0) = 1 \) and \( \dot{x}_1(0) = 0 \). Find another solution \( x_2 \) such that \( x_2(0) = 0 \) and \( \dot{x}_2(0) = 1 \).

We can assume \( a \geq 0 \). From Question 5 we know that both \( x(t) = e^{at} \) and \( x(t) = e^{-at} \) are solutions to \( \ddot{x} - a^2 x = 0 \). Then for any constants \( c_1 \) and \( c_2 \), \( x(t) = c_1 e^{at} + c_2 e^{-at} \) is also a solution to \( \ddot{x} - a^2 x = 0 \), with \( x(0) = c_1 + c_2 \) and \( \dot{x}(0) = a(c_1 - c_2) \). So, for \( a > 0 \), \( x_1(t) \) must satisfy \( c_1 + c_2 = 1 \) and \( a(c_1 - c_2) = 0 \), which implies
\[ c_1 = c_2 = 1/2. \] So
\[ x_1(t) = \frac{1}{2}e^{at} + \frac{1}{2}e^{-at} = \cosh(at). \]

For \( x_2(t) \), need \( c_1 + c_2 = 0 \) and \( a \left( c_1 - c_2 \right) = 1 \), so \( c_1 = -c_2 = \frac{1}{2a} \) and
\[ x_2(t) = \frac{1}{2a}e^{at} - \frac{1}{2a}e^{-at} = \frac{1}{a} \sinh(at). \]

If \( a = 0 \), \( x_1(t) = 1 \) and \( x_2 \) does not exist.