Polynomial Input: The Method of Undetermined Coefficients

1. The Basic Result

A polynomial is a function of the form

\[ q(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0. \]

The largest \( k \) for which \( a_k \neq 0 \) is the degree of \( q(x) \). (The zero function is a polynomial too, but it doesn’t have a degree.)

Note that \( q(0) = a_0 \) and \( q'(0) = a_1 \).

Here is the basic fact about the response of an LTI system with characteristic polynomial \( p(s) \) to polynomial signals:

**Theorem. (Undetermined coefficients)** If \( p(0) \neq 0 \), and \( q(x) \) is a polynomial of degree \( n \), then

\[ p(D)y = q(x) \]

has exactly one solution which is a polynomial, and it is of degree \( n \).

The best way to see this, and to see how to compute this polynomial particular solution, is by examples.

2. The Method of Undetermined Coefficients

Given the linear time invariant (LTI) DE \( p(D)y = q(x) \) with \( q(x) \) is a polynomial of degree \( n \), the Undetermined Coefficient (UC) solution method, as we discussed in the previous note, is to assume a particular solution of the form \( y_p = h(x) \), where \( h(x) \) is a polynomial of degree \( n \) with unknown (“undetermined”) coefficients, and then to find the coefficients by substituting \( y_p \) into the ODE. It’s important to do the work systematically; we suggest following the format given in the following example.

**Example 1.** Find a particular solution \( y_p \) to \( y'' + 3y' + 4y = 4x^2 - 2x \).

**Solution.** Our trial solution is \( y_p = Ax^2 + Bx + C \); we format the work as follows. The lines show the successive derivatives; multiply each line by the factor given in the ODE, and add the equations, collecting like powers of \( x \) as you go. The fourth line shows the result; the sum on the left takes into account that \( y_p \) is supposed to be a particular solution to the given
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ODE.

\[ \times 4 \quad y_p = Ax^2 + Bx + C \]
\[ \times 3 \quad y_p' = 2Ax + B \]
\[ y_p'' = 2A \]
\[ 4x^2 - 2x = (4A)x^2 + (4B + 6A)x + (4C + 3B + 2A). \]

Equating like powers of \( x \) in the last line gives the three equations

- \( 4A = 4 \)
- \( 4B + 6A = -2 \)
- \( 4C + 3B + 2A = 0 \)

solving them in order gives \( A = 1, B = -2, C = 1 \), so that \( y_p = x^2 - 2x + 1 \).

**Example 2.** Solve \( y'' + 5y' + 4y = 2x + 3 \).

**Solution.** Guess a trial solution of the form \( y_p = Ax + B \) (same degree as input).

Substitute in DE: \( y_p'' + 5y_p' + 4y_p = 0 + 5(A) + 4(Ax + B) = 2x + 3 \).

\[ \Rightarrow 4Ax + (5A + 4B) = 2x + 3. \]

Equate coefficients: \( 4A = 2, \quad 5A + 4B = 3 \).

Triangular system is easy to solve: \( A = 1/2, B = 1/8 \).

Find solution of homogeneous DE: \( y'' + 5y' + 4y = 0 \)

Char. equation: \( r^2 + 5r + 4 = 0 \) \Rightarrow \( r = -1, -4 \)

\[ \Rightarrow y_h = c_1e^{-t} + c_2e^{-4t} \]

\[ \Rightarrow \text{general solution to DE} = y = y_p + y_h. \]

**Example 3.** Solve \( y'' + 5y' + 4y = x^2 + 3x \)

**Solution.** Guess a trial solution of the form \( y_p = Ax^2 + Bx + C \) (same degree as input).

Substitute this into the DE:

\[ y'' + 5y' + 4y = 2A + 5(2Ax + B) + 4(Ax^2 + Bx + C) = x^2 + 3x \]

\[ \Rightarrow 4Ax^2 + (10A + 4B)x + (2A + 5B + 4C) = x^2 + 3x \]

Equate coefficients: \( 4A = 1, \quad 10A + 4B = 3, \quad 2A + 5B + 4C = 0 \)

Triangular system is easy to solve: \( A = 1/4, B = 1/8, C = -9/32 \)

\[ \Rightarrow y_p = \frac{1}{4}x^2 + \frac{1}{8}x - \frac{9}{32}. \]

Use homogeneous solution from previous example to get the general solution to DE: \( y = y_p + y_h. \)