1. A certain periodic function has Fourier series

\[ f(t) = 1 + \frac{\cos(\pi t)}{2} + \frac{\cos(2\pi t)}{4} + \frac{\cos(3\pi t)}{8} + \frac{\cos(4\pi t)}{16} + \cdots \]

(a) What is the minimal period of \( f(t) \)? \[4\]

(b) Is \( f(t) \) even, odd, neither, or both? \[4\]

(c) Please give the Fourier series of a periodic solution (if one exists) of

\[ \ddot{x} + \omega_n^2 x = f(t) \]

\[8\]

(d) For what values of \( \omega_n \) is there no periodic solution? \[4\]
2. Let $f(t) = (u(t + 1) - u(t - 1))t$.

(a) Sketch a graph of $f(t)$.  

(b) Sketch a graph of the generalized derivative $f'(t)$.  

(c) Write a formula for the generalized derivative $f'(t)$. 

3. Let \( p(D) \) be the operator whose unit impulse response is given by \( w(t) = e^{-t} - e^{-3t} \).

(a) Using convolution, find the unit step response of this operator: the solution to \( p(D)v = \) \( u(t) \) with rest initial conditions.

(b) What is the transfer function \( W(s) \) of the operator \( p(D) \)?

(c) What is the characteristic polynomial \( p(s) \)?
4 (a) Find a generalized function $f(t)$ with Laplace transform $F(s) = \frac{e^{-s}(s - 1)}{s}$. [10]

(b) Find a function $f(t)$ with Laplace transform $F(s) = \frac{s + 10}{s^3 + 2s^2 + 10s}$. [10]
5. Let $W(s) = \frac{s + 10}{s^3 + 2s^2 + 10s}$.

(a) Sketch the pole diagram of $W(s)$. [10]

(b) If $W(s)$ is the transfer function of an LTI system, what is the Laplace transform of the response from rest initial conditions to the input $\sin(2t)$? [10]
Properties of the Laplace transform

0. Definition: \[ \mathcal{L}[f(t)] = F(s) = \int_{0^-}^{\infty} f(t)e^{-st} \, dt \quad \text{for Re } s > 0. \]

1. Linearity: \[ \mathcal{L}[af(t) + bg(t)] = aF(s) + bG(s). \]

2. Inverse transform: \[ F(s) \text{ essentially determines } f(t). \]

3. s-shift rule: \[ \mathcal{L}[e^{at}f(t)] = F(s-a). \]

4. t-shift rule: \[ \mathcal{L}[f_a(t)] = e^{-as}F(s), \quad f_a(t) = u(t-a)f(t-a) = \begin{cases} f(t-a) & \text{if } t > a \\ 0 & \text{if } t < a \end{cases}. \]

5. s-derivative rule: \[ \mathcal{L}[tf(t)] = -F'(s). \]

6. t-derivative rule: \[ \mathcal{L}[f'(t)] = sF(s) - f(0^-), \text{ where } f'(t) \text{ denotes the generalized derivative.} \]

7. Convolution rule: \[ \mathcal{L}[f(t) * g(t)] = F(s)G(s), \quad f(t) * g(t) = \int_{0^-}^{t^+} f(t-\tau)g(\tau) \, d\tau. \]

8. Weight function: \[ \mathcal{L}[w(t)] = W(s) = 1/p(s), \text{ } w(t) \text{ the unit impulse response.} \]

Formulas for the Laplace transform

\[ \mathcal{L}[1] = \frac{1}{s}, \quad \mathcal{L}[e^{at}] = \frac{1}{s-a}, \quad \mathcal{L}[t^n] = \frac{n!}{s^{n+1}} \]

\[ \mathcal{L}[\cos(\omega t)] = \frac{s}{s^2 + \omega^2}, \quad \mathcal{L}[\sin(\omega t)] = \frac{\omega}{s^2 + \omega^2} \]

\[ \mathcal{L}[t \cos(\omega t)] = \frac{2\omega s}{(s^2 + \omega^2)^2}, \quad \mathcal{L}[t \sin(\omega t)] = \frac{s^2 - \omega^2}{(s^2 + \omega^2)^2} \]

Fourier coefficients for periodic functions of period 2\pi:

\[ f(t) = \frac{a_0}{2} + a_1 \cos(t) + a_2 \cos(2t) + \cdots + b_1 \sin(t) + b_2 \sin(2t) + \cdots \]

\[ a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(mt) \, dt, \quad b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(mt) \, dt \]

If sq(t) is the odd function of period 2\pi which has value 1 between 0 and \pi, then

\[ sq(t) = \frac{4}{\pi} \left( \sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \cdots \right) \]