More Entries for the Laplace Table

In this note we will add some new entries to the table of Laplace transforms.

1. \( L(\cos(\omega t)) = \frac{s}{s^2 + \omega^2} \), with region of convergence \( \text{Re}(s) > 0 \).

2. \( L(\sin(\omega t)) = \frac{\omega}{s^2 + \omega^2} \), with region of convergence \( \text{Re}(s) > 0 \).

**Proof:** We already know that \( L(e^{at}) = \frac{1}{s-a} \). Using this and Euler’s formula for the complex exponential, we obtain

\[
L(\cos(\omega t) + i\sin(\omega t)) = L(e^{i\omega t}) = \frac{1}{s - i\omega} = \frac{1}{s - i\omega} \cdot \frac{s + i\omega}{s + i\omega} = \frac{s + i\omega}{s^2 + \omega^2}.
\]

Taking the real and imaginary parts gives us the formulas.

\[
L(\cos(\omega t)) = \text{Re} \left( L(e^{i\omega t}) \right) = s/(s^2 + \omega^2) \\
L(\sin(\omega t)) = \text{Im} \left( L(e^{i\omega t}) \right) = \omega/(s^2 + \omega^2)
\]

The region of convergence follow from the fact that \( \cos(\omega t) \) and \( \sin(\omega t) \) both have exponential order 0.

Another approach would have been to use integration by parts to compute the transforms directly from the Laplace integral.

3. For a positive integer \( n \), \( L(t^n) = n!/s^{n+1} \). The region of convergence is \( \text{Re}(s) > 0 \).

**Proof:** We start with \( n = 1 \).

\[
L(t) = \int_0^\infty te^{-st} \, dt
\]

Using integration by parts:

\[
\begin{align*}
    u &= t & du &= e^{-st} \\
    dv &= e^{-st} & v &= e^{-st} / (-s)
\end{align*}
\]

\[
L(t) = \left. \left(-\frac{te^{-st}}{s} \right) \right|_0^\infty + \frac{1}{s} \int_0^\infty e^{-st} \, dt.
\]

For \( \text{Re}(s) > 0 \) the first term is 0 and the second term is \( \frac{1}{s} L(1) = 1/s^2 \). Thus, \( L(t) = 1/s^2 \).

Next let’s do \( n = 2 \):

\[
L(t^2) = \int_0^\infty t^2 e^{-st} \, dt
\]
Again using integration by parts:

\[ u = t^2 \quad dv = e^{-st} \]

\[ du = 2t \quad v = e^{-st} / (-s) \]

\[ \mathcal{L}(t^2) = -\frac{t^2 e^{-st}}{s} \bigg|_0^\infty + \frac{1}{s} \int_0^\infty 2te^{-st} \, dt. \]

For \( \text{Re}(s) > 0 \) the first term is 0 and the second term is \( \frac{1}{s} \mathcal{L}(2t) = 2/s^3 \).

Thus, \( \mathcal{L}(t^2) = 2/s^3 \).

We can see the pattern: there is a reduction formula for

\[ \mathcal{L}(t^n) = \int_0^\infty t^n e^{-st} \, dt. \]

Integration by parts:

\[ u = t^n \quad dv = e^{-st} \]

\[ du = nt^{n-1} \quad v = e^{-st} / (-s) \]

\[ \mathcal{L}(t^n) = -\frac{t^ne^{-st}}{s} \bigg|_0^\infty + \frac{1}{s} \int_0^\infty nt^{n-1} e^{-st} \, dt. \]

For \( \text{Re}(s) > 0 \) the first term is 0 and the second term is \( \frac{1}{s} \mathcal{L}(nt^{n-1}) \).

Thus, \( \mathcal{L}(t^n) = \frac{n}{s} \mathcal{L}(t^{n-1}) \).

Thus we have

\[ \mathcal{L}(t^3) = \frac{3}{s} \mathcal{L}(t^2) = \frac{3 \cdot 2}{s^3} = \frac{3!}{s^3} \]

\[ \mathcal{L}(t^4) = \frac{4}{s} \mathcal{L}(t^3) = \frac{4 \cdot 3!}{s^5} = \frac{4!}{s^5} \]

\[ \ldots \]

\[ \mathcal{L}(t^n) = \frac{n!}{s^{n+1}}. \]

4. (s-shift formula) If \( z \) is any complex number and \( f(t) \) is any function then

\[ \mathcal{L}(e^{zt} f(t)) = F(s - z). \]

As usual we write \( F(s) = \mathcal{L}(f)(s) \). If the region of convergence for \( \mathcal{L}(f) \) is \( \text{Re}(s) > a \) then the region of convergence for \( \mathcal{L}(e^{zt} f(t)) \) is \( \text{Re}(s) > \text{Re}(z) + a \).

**Proof:** We simply calculate

\[ \mathcal{L}(e^{zt} f(t)) = \int_0^\infty e^{zt} f(t)e^{-st} \, dt \]

\[ = \int_0^\infty f(t)e^{-(s-z)t} \, dt \]

\[ = F(s - z). \]
Example. Find the Laplace transform of $e^{-t} \cos(3t)$.

Solution. We could do this by using Euler’s formula to write

$$e^{-t} \cos(3t) = (1/2) \left( e^{(-1+3i)t} + e^{(-1-3i)t} \right)$$

but it’s even easier to use the $s$-shift formula with $z = -1$, which gives

$$\mathcal{L}(e^{-t} f(t)) = F(s + 1),$$

where here $f(t) = \cos(3t)$, so that $F(s) = s/(s^2 + 9)$. Shifting $s$ by -1 according to the $s$-shift formula gives

$$\mathcal{L}(e^{-t} \cos(3t)) = F(s + 1) = \frac{s + 1}{(s + 1)^2 + 9}.$$

We record two important cases of the $s$-shift formula:

4a) $\mathcal{L}(e^{zt} \cos(\omega t)) = \frac{s - z}{(s - z)^2 + \omega^2}$

4b) $\mathcal{L}(e^{zt} \sin(\omega t)) = \frac{\omega}{(s - z)^2 + \omega^2}$.

Consistency.

It is always useful to check for consistency among our various formulas:

1. We have $\mathcal{L}(1) = 1/s$, so the $s$-shift formula gives $\mathcal{L}(e^{zt} \cdot 1) = 1/(s - z)$. This matches our formula for $\mathcal{L}(e^{zt})$.

2. We have $\mathcal{L}(t^n) = n!/s^{n+1}$. If $n = 1$ we have $\mathcal{L}(t^0) = 0!/s = 1/s$. This matches our formula for $\mathcal{L}(1)$. 