Part II Problems and Solutions

Problem 1: [Periodic solutions] Let \( g(t) \) be the function which is periodic of period \( 2\pi \), and such that \( g(t) = t \) for \( -\frac{\pi}{2} \leq t \leq \frac{\pi}{2} \) and \( g(t) = \pi - t \) for \( \frac{\pi}{2} \leq t \leq \frac{3\pi}{2} \).

(a) Find a periodic solution to \( \ddot{x} + \omega_0^2 x = g(t) \) (if there is one).

(b) For what (positive) values of \( \omega_0 \) are there no periodic solution?

(c) Write \( \omega_r \) for the smallest number you found in (b). For \( \omega_0 \) just less than \( \omega_r \), what is the solution like, approximately? How about for \( \omega_0 \) just larger than \( \omega_r \)?

(d) For what values of \( \omega_0 \) are there more than one periodic solution?

(e) For the values of \( \omega_0 \) found in (d), are all solutions to \( \ddot{x} + \omega_0^2 x = g(t) \) periodic?

Solution: (a) Using the integral formulas for the coefficients as usual we first calculate the Fourier series for \( g(t) \). The result is

\[
g(t) = \frac{4}{\pi} \left( \sin(t) - \frac{1}{3^2} \sin(3t) + \frac{1}{5^2} \sin(5t) - \cdots \right).\]

By superposition and the fact that

\[
\ddot{x} + \omega_0^2 x = A \sin(\omega t) \quad \text{has solution} \quad A \frac{\sin(\omega t)}{\omega_0^2 - \omega^2},
\]

we find that a solution to \( \ddot{x} + \omega_0^2 x = g(t) \) is given by

\[
x_p = \frac{4}{\pi} \left( \frac{\sin(t)}{\omega_0^2 - 1^2} - \frac{1}{3^2} \frac{\sin(3t)}{\omega_0^2 - 3^2} + \cdots \right),
\]

as long as \( \omega_0 \) is not an odd integer.

(b) If \( \omega_0 \) is an odd integer there is no periodic solution.

(c) \( \omega_r = 1 \). For \( \omega \) just less than 1, the term \( \frac{4}{\pi} \frac{\sin(t)}{\omega_0^2 - 1^2} \) dominates, and \( x_p \) is relatively close to this: This is antiphase with \( \sin(t) \) and has large amplitude. When \( \omega_0 \) is just greater than 1, the same term occurs and dominates but now is a positive multiple of \( \sin(t) \), so the system response is in phase with the input.

(d) This is a tricky question. When \( \omega_0 \) is not an odd integer, the solution \( x_p \) above is periodic of period \( 2\pi \). The general solution of the homogeneous equation is \( a \cos(\omega_0 t) + b \sin(\omega_0 t) \), which is periodic of period \( \frac{2\pi}{\omega_0} \). The sum is periodic if some multiple of \( 2\pi \) is
equal to some multiple of $\frac{2\pi}{\omega_0}$, and this happens when $\omega_0$ is a rational number (but not an odd integer).

(e) Yes. [They are periodic of period $2\pi$ if $\omega_0$ is an even integer.]
18.03SC Differential Equations
Fall 2011

For information about citing these materials or our Terms of Use, visit: http://ocw.mit.edu/terms.