Compute a Fourier Series

Exercise. We warm up with a reminder of how one computes the Fourier series of a given periodic function using the integral Fourier coefficient formulas.

Compute the Fourier series for the period $2\pi$ continuous sawtooth function $f(t) = |t|$ for $-\pi \leq t \leq \pi$.

Answer.

Figure 1. Graph of the period $2\pi$ continuous sawtooth function.

The period is $2\pi$, so the half-period $L = \pi$. Since $f(t) = |t|$ for $-\pi \leq t \leq \pi$, it is an even function we know the Fourier sine coefficients $b_n$ must be zero.

Computing the cosine coefficients we get: For $n \neq 0$:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |t| \cos(nt) \, dt = \frac{2}{\pi} \int_{0}^{\pi} t \cos(nt) \, dt$$

$$= \frac{2}{n} \left( \frac{t \sin(nt)}{n} + \frac{\cos(nt)}{n^2} \right) \bigg|_{0}^{\pi} = \frac{2}{n^2 \pi}((-1)^n - 1) = \left\{ \begin{array}{ll} -\frac{4}{n^2 \pi} & \text{for } n \text{ odd} \\ 0 & \text{for even} \end{array} \right.$$  

For $n = 0$:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} |t| \, dt = \frac{2}{\pi} \int_{0}^{\pi} t \, dt = \pi.$$

Thus, $f(t)$ has Fourier series

$$f(t) = \frac{\pi}{2} - \frac{4}{\pi} \left( \cos t + \frac{\cos(3t)}{3^2} + \frac{\cos(5t)}{5^2} + \cdots \right)$$

$$= \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\cos(nt)}{n^2}$$