Scaling and Shifting

There is a very useful class of shortcuts which allows us to use the known Fourier series of a function \( f(t) \) to get the series for a function related to \( f(t) \) by shifts and scale changes. We illustrate this technique with a collection of examples of related functions.

We let \( \text{sq}(t) \) be the standard odd, period \( 2\pi \) square wave.

\[
\text{sq}(t) = \begin{cases} 
-1 & \text{for } -\pi \leq t < 0 \\
1 & \text{for } 0 \leq t < \pi 
\end{cases}
\]  

\[\text{Figure 0: The graph of } \text{sq}(t), \text{ the odd, period } 2\pi \text{ square wave.}\]

We already know the Fourier series for \( \text{sq}(t) \). It is

\[
\text{sq}(t) = \frac{4}{\pi} \left( \sin(t) + \frac{1}{3} \sin(3t) + \frac{1}{5} \sin(5t) + \cdots \right) = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin(nt)}{n}
\]  

1. Shifting and Scaling in the Vertical Direction

Example 1. (Shifting) Find the Fourier series of the function \( f_1(t) \) whose graph is shown.

\[\text{Figure 1: } f_1(t) = \text{sq}(t) \text{ shifted up by 1 unit.}\]

Solution. The graph in Figure 1 is simply the graph in Figure 0 shifted upwards one unit. That is, \( f_1(t) = 1 + \text{sq}(t) \). Therefore

\[
f_1(t) = 1 + \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin(nt)}{n}.
\]

Example 2. (Scaling) Let \( f_2(t) = 2 \text{sq}(t) \). Sketch its graph and find its Fourier series.
Solution.

![Graph of \( f_2(t) = 2 \text{sq}(t) \).]

The Fourier series of \( f_2(t) \) comes from that of \( \text{sq}(t) \) by multiplying by 2.

\[
f_2(t) = \frac{8}{\pi} \sum_{n \text{ odd}} \frac{\sin(nt)}{n}.
\]

**Example 3.** We can combine shifting and scaling along the vertical axis. Let \( f_3(t) \) be the function shown in Figure 3. Write it in terms of \( \text{sq}(t) \) and find its Fourier series.

![Graph of \( f_3(t) \).]

**Solution.** \( f_3(t) = \frac{1}{2} (1 + \text{sq}(t)) = \frac{1}{2} + \frac{2}{\pi} \sum_{n \text{ odd}} \frac{\sin nt}{n} \).

2. **Scaling and Shifting in \( t \)**

**Example 4. (Scaling in time)** Find the Fourier series of the function \( f_4(t) \) whose graph is shown.

![Graph of \( \text{sq}(t) \) scaled in time.]

In Figure 4 the point marked 1 on the \( t \)-axis corresponds with the point marked \( \pi \) in Figure 0. This shows that \( f_4(t) = \text{sq}(\pi t) \) and therefore we replace \( t \) by \( \pi t \) in the Fourier series of \( \text{sq}(t) \).

\[
f_4(t) = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin(n\pi t)}{n}.
\]
Example 5. (Shifting in time) Let \( f_5(t) = \text{sq}(t + \pi/2) \). Graph this function and find its Fourier series.

Solution. We have \( f_5(t) \) is \( \text{sq}(t) \) shifted to the left by \( \pi/2 \). Therefore

\[
 f_5(t) = \frac{4}{\pi} \left( \sin(t + \pi/2) + \frac{\sin(3t + 3\pi/2)}{3} + \ldots \right) = \frac{4}{\pi} \left( \cos t - \frac{\cos 3t}{3} + \ldots \right)
\]

(To simplify the series we used the trig identities \( \sin(\theta + \pi/2) = \cos(\theta) \) and \( \sin(\theta + 3\pi/2) = -\cos(\theta) \) etc.)

![Figure 5: \( \text{sq}(t) \) shifted in time.](image)

Notice that \( f_5(t) \) is even, and so must have only cosine terms in its series, which is in fact confirmed by the simplified form above.