Integration and Differentiation

We can integrate a Fourier series term-by-term:

**Example 1.** Let

\[ f(t) = 1 + \cos t + \frac{\cos 2t}{2} + \frac{\cos 3t}{3} + \ldots \]

then,

\[ h(t) = \int_0^t f(u) \, du = t + \sin t + \frac{\sin 2t}{2} + \frac{\sin 3t}{3} + \ldots \]

**Note:** The integrated function \( h(t) \) is not periodic (because of the \( t \) term), so the result is a series, but not a Fourier series.

We can also differentiate a Fourier series term-by-term to get the Fourier series of the derivative function.

**Example 2.** Let \( f(t) \) be the period \( 2\pi \) triangle wave (continuous sawtooth) given on the interval \([−\pi, \pi)\) by \( f(t) = |t| \). Its Fourier series is

\[ f(t) = \frac{\pi}{2} - \frac{4}{\pi} \left( \cos t + \frac{\cos 3t}{3^2} + \frac{\cos 5t}{5^2} + \ldots \right) \]

In the previous session we computed the Fourier series of a period 2 triangle wave. This series can then be obtained from that one by scaling by \( \pi \) in both time and the vertical dimension, using the methods we learned in the previous note.

The derivative of \( f(t) \) is the square wave. (You should verify this). Differentiating the Fourier series of \( f(t) \) term-by-term gives

\[ f'(t) = \frac{4}{\pi} \left( \sin t + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \ldots \right), \]

which is, indeed, the Fourier series of the period \( 2\pi \) square wave we found in the previous session.

![Figure 1: The period 2\pi triangle wave.](image)
Example 3. What happens if you try to differentiate the square wave

\[ \text{sq}(t) = \frac{4}{\pi} \left( \sin t + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \ldots \right) \]

Solution. Differentiation term-by-term gives

\[ \text{sq}'(t) = \frac{4}{\pi} \left( \cos t + \cos 3t + \cos 5t + \ldots \right) \]

But, what is meant by \( \text{sq}'(t) \)? Since \( \text{sq}(t) \) consists of horizontal segments its derivative at most places is 0. However we can’t ignore the ‘vertical’ segments where the function has a jump discontinuity. For now, the best we can say is that the slope is infinite at these jumps and \( \text{sq}'(t) \) doesn’t exist. Later in this unit we will learn about delta functions and generalized derivatives, which will allow us to make better sense of \( \text{sq}'(t) \).