Part I Problems and Solutions

Problem 1: Compute the following matrix products:

a) \( \begin{bmatrix} 1 & 2 \\ x & y \end{bmatrix} \)

b) \( \begin{bmatrix} 1 \\ 2 \\ x & y \end{bmatrix} \)

c) \( \begin{bmatrix} a & b \\ c & d \\ x & y \end{bmatrix} \)

d) \( \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ x & u \\ y & v \end{bmatrix} \)

Solution:

a) \( x + 2y \)

b) \( x + y \\
    2x + 2y \)

c) \( ax + by \\
    cx + dy \)

d) \( \begin{bmatrix} x + 2y & u + 2v \\ 3x + 4y & 3u + 4v \end{bmatrix} \)

Problem 2: Let \( A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \) and \( B = \begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix} \). Show that \( AB \neq BA \).

Solution:

\( AB = \begin{bmatrix} 4 & 1 \\ -2 & -4 \end{bmatrix} \)

\( BA = \begin{bmatrix} -3 & 1 \\ 5 & 3 \end{bmatrix} \)

Problem 3: Write the following equations as equivalent first-order systems.
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a) \( \frac{d^2x}{dt^2} + 5\frac{dx}{dt} + tx^2 = 0 \)

b) \( y'' - x^2y' + (1 - x^2)y = \sin x \)

Solution:

a) \( x'' + 5x' + tx^2 = 0 \rightarrow x' = y, \ y' = -tx^2 - 5y \)

b) \( y'' - x^2y' + (1 - x^2)y = \sin x \rightarrow y' = z, \ z' = (x^2 - 1)y + x^2z + \sin x \)

Problem 4: Solve the system \( x' = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x \) in two ways:

a) Solve the second equation, substitute for \( y \) in the first equation, and solve it.

b) Eliminate \( y \) by solving the first equation for \( y \), then substitute into the second equation, getting a second order equation for \( x \). Solve it, and then find \( y \) from the first equation. Do your two methods give the same answer?

Solution:

\[ \begin{bmatrix} x' \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \]

or \( x' = x + y, \ y' = y \).

a) From the second equation, \( y = c_1e^t \), so \( x' - x = c_1e^t \), so the solution is \( x = c_2e^t + c_1te^t, \ y = c_1e^t \).

b) Here we eliminate \( y \) instead. \( y = x' - x \) so \( (x' - x)' = x'' - 2x' + x = 0 \rightarrow (m - 1)^2 = 0 \) (char. eqn.). Thus, we have \( x = c_1e^t + c_2te^t, \ y = c_2e^t \) (since \( y = x' - x \)). This is the same as before, with \( c_1, c_2 \) switched.