Review of Vectors and Matrices

1. Vectors

A vector (or n-vector) is an n-tuple of numbers; they are usually real numbers, but we will sometimes allow them to be complex numbers. All the rules and operations below apply just as well to n-tuples of complex numbers. (In the context of vectors, a single real or complex number, i.e., a constant, is called a scalar.) As we are dealing with 2 × 2 linear systems, we are primarily interested in scalars and 2-vectors: ordered pairs of numbers.

The pair can be written horizontally as a row vector or vertically as a column vector. In these notes, it will almost always be a column. To save space, we will sometimes write the column vector as shown below; the small T stands for transpose, and means: change the row to a column.

\[
a = (a, b) \quad \text{row vector} \quad a = (a, b)^T \quad \text{column vector}
\]

These notes use boldface for vectors hope; in handwriting, place an arrow \( \vec{a} \) over the letter.

Vector operations. Here are two standard operations on vectors:

- addition: \((a, b) + (c, d) = (a + c, b + d)\).
- multiplication by a scalar: \(c(a, b) = (ca, cb)\)
- scalar product: \((a, b)(c, d) = ac + bd\)

2. Matrices

An \(m \times n\) matrix \(A\) is a rectangular array of numbers (real or complex) having \(m\) rows and \(n\) columns. The element in the \(i\)-th row and \(j\)-th column is called the \(ij\)-th entry and written \(a_{ij}\). The matrix itself is sometimes written \((a_{ij})\), i.e., by giving its generic entry, inside the matrix parentheses. We will be interested in matrices where \(m\) and \(n\) are at most 2.

Note that a \(1 \times 2\) matrix is a row vector; an \(2 \times 1\) matrix is a column vector.

Matrix operations.

- addition: if \(A\) and \(B\) are both \(m \times n\) matrices, they are added by adding the corresponding entries; i.e., if \(A = (a_{ij})\) and \(B = (b_{ij})\), then \(A + B = (a_{ij} + b_{ij})\).
multiplication by a scalar: to get $cA$, multiply every entry of $A$ by the scalar $c$; i.e., if $A = (a_{ij})$, then $cA = (ca_{ij})$.

matrix multiplication: if $A$ is an $m \times n$ matrix and $B$ is an $n \times k$ matrix, their product $AB$ is an $m \times k$ matrix, defined by using the scalar product operation:

$$ij\text{-th entry of } AB = (i\text{-th row of } A)(j\text{-th column of } B)^T$$

where the scalar product of two 1-vectors is just their normal product.

The definition makes sense since both vectors on the right are vectors of the same length $n$. In what follows, the most important cases of matrix multiplication will be:
(i) $A$ and $B$ are square $2 \times 2$ matrices. In this case, multiplication is always possible, and the product $AB$ is again an $2 \times 2$ matrix.
(ii) $A$ is an $2 \times 2$ matrix and $B = b$, a column 2-vector. In this case, the matrix product $Ab$ is again a column 2-vector.

Laws satisfied by the matrix operations.
For any matrices for which the products and sums below are defined, we have

$$(A B) C = A (B C) \quad \text{(associative law)}$$
$$(A + B) C = AC + AC \quad \text{(distributive laws)}$$

$$(A B) \neq B A \quad \text{(commutative law fails in general)}$$

The identity matrix $I$ is the $2 \times 2$ matrix with 1’s on the main diagonal (upper left and bottom right), and 0’s elsewhere. If $A$ is an arbitrary $2 \times 2$ matrix, it is easy to check from the definition of matrix multiplication that

$$AI = A \quad \text{and} \quad IA = A.$$

The exercises later in this session should help you get familiar with all these concepts.
18.03SC Differential Equations
Fall 2011

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