Describing a First order System Using Matrix Notation

1. Description of the Equation

A general $2 \times 2$ linear system is given by:

\[
\begin{align*}
\dot{x} &= ax + by \\
\dot{y} &= cx + dy
\end{align*}
\]

The terms have been arranged in a suggestive manner. We can express this system using matrices and vectors:

\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} =
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}.
\]

We can present this in the following even more compact form.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and write $u$ for the column vector $\begin{pmatrix} x \\ y \end{pmatrix}$.

We have $\dot{u}(t) = \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix}$ and the system is simply $\dot{u} = Au$.

Example 1. Our favorite system, governing the rabbit populations in farmers Jones’ and McGregor’s fields, was

\[
\begin{align*}
\dot{x} &= 0.3x + 0.1y \\
\dot{y} &= 0.2x + 0.4y,
\end{align*}
\]

which has matrix form

\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} =
\begin{pmatrix}
0.3 & 0.1 \\
0.2 & 0.4
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix} \quad \text{or} \quad \dot{u} = Au, \quad \text{where} \quad A = \begin{pmatrix} 0.3 & 0.1 \\ 0.2 & 0.4 \end{pmatrix}
\]

2. Description of the Solution

To describe the solution, we will use the column vector $u(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$.

Example 2. Earlier we used the method of elimination to solve the system in example 1. We found $x(t) = c_1e^{0.5t} + c_2e^{0.2t}$, $y(t) = 2c_1e^{0.5t} - c_2e^{0.2t}$.

Rewriting this in vector form we have

\[
u(t) = \begin{pmatrix} c_1e^{0.5t} + c_2e^{0.2t} \\ 2c_1e^{0.5t} - c_2e^{0.2t} \end{pmatrix}.
\]
We can rewrite this as

\[ u(t) = c_1 e^{0.5t \begin{pmatrix} 1 \\ 2 \end{pmatrix}} + c_2 e^{0.2t \begin{pmatrix} 1 \\ -1 \end{pmatrix}}, \]

which is a clearer way of presenting it. Let

\[ u_1(t) = e^{5t \begin{pmatrix} 1 \\ 2 \end{pmatrix}} \quad \text{and} \quad u_2(t) = e^{2t \begin{pmatrix} 1 \\ -1 \end{pmatrix}}. \]

The column vectors \( u_1(t) \) and \( u_2(t) \) are both solutions. Since they both involve only one form of exponential, they are sometimes known as basic independent solutions, or normal modes. The general solution is a linear combination of them. We will learn much more about normal modes in the sessions on matrix methods and the phase portrait.

**Remark.** As with linear second order ODEs in unit 2, the general solution to to a 2 linear system should always consist of linear combinations of two truly different solutions. It is not necessary, but usually our techniques will make these two solutions the normal modes.

3. **Geometry of the Solutions**

Suppose you want to plot a solution \( u(t) \). As time increases, it traces a curve in the \( xy \)-plane.

**Example.** The solution \( u_1(t) \) traces a ray that passes through \((1, 2)\) at \( t = 0 \) and move off towards infinity in a straight line, with exponential speed. There is another ray through \((1, -1)\), corresponding to the solution \( u_2(t) \). (This is tricky: the exponential in the formula for \( u_1 \) might make you think the trajectory is curved. However, if you look carefully at the formula you will see \( u_1(t) \) is always a multiple of the vector \((1, 2)^T\).)

The applet *Linear phase portrait: matrix entries* will allow us to visualise this nicely, and get a feel for other sorts of trajectories. Later in this session you will look at this applet.