Describing a First order System Using Matrix Notation

1. Description of the Equation

A general 2×2 linear system is given by:

$$\dot{x} = ax + by$$
$$\dot{y} = cx + dy$$

The terms have been arranged in a suggestive manner. We can express this system using matrices and vectors:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

We can present this in the following even more compact form.
Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and write $\mathbf{u}$ for the column vector $\begin{pmatrix} x \\ y \end{pmatrix}$.
We have $\dot{\mathbf{u}}(\mathbf{t}) = \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix}$ and the system is simply $\dot{\mathbf{u}} = A\mathbf{u}$.

Example 1. Our favorite system, governing the rabbit populations in farmers Jones' and McGregor's fields, was

$$\dot{x} = 0.3x + 0.1y$$
$$\dot{y} = 0.2x + 0.4y,$$

which has matrix form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0.3 & 0.1 \\ 0.2 & 0.4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{or} \quad \dot{\mathbf{u}} = A\mathbf{u}, \quad \text{where} \quad A = \begin{pmatrix} 0.3 & 0.1 \\ 0.2 & 0.4 \end{pmatrix}$$

2. Description of the Solution

To describe the solution, we will use the column vector $\mathbf{u}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$.

Example 2. Earlier we used the method of elimination to solve the system in example 1. We found $x(t) = c_1 e^{0.5t} + c_2 e^{0.2t}$, $y(t) = 2c_1 e^{0.5t} - c_2 e^{0.2t}$. Rewriting this in vector form we have

$$\mathbf{u}(t) = \begin{pmatrix} c_1 e^{0.5t} + c_2 e^{0.2t} \\ 2c_1 e^{0.5t} - c_2 e^{0.2t} \end{pmatrix}.$$

We can rewrite this as

$$\mathbf{u}(t) = c_1 e^{0.5t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{0.2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

which is a clearer way of presenting it. Let

$$\mathbf{u}_1(t) = e^{.5t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{and } \mathbf{u}_2(t) = e^{.2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The column vectors $\mathbf{u}_1(t)$ and $\mathbf{u}_2(t)$ are both **solutions**. Since they both involve only one form of exponential, they are sometimes known as **basic independent solutions**, or **normal modes**. The general solution is a linear combination of them. We will learn much more about normal modes in the sessions on matrix methods and the phase portrait.

Remark. As with linear second order ODE's in unit 2, the general solution to to a 2 linear system should always consist of linear combinations of two truly different solutions. It is not necessary , but usually our techniques will make these two solutions the normal modes.

3. Geometry of the Solutions

Suppose you want to plot a solution $\mathbf{u}(t)$. As time increases, it traces a curve in the xy-plane.

Example. The solution $\mathbf{u}_1(t)$ traces a ray that passes through $(1, 2)$ at $t = 0$ and move off towards infinity in a straight line, with exponential speed. There is another ray through $(1, -1)$, corresponding to the solution $\mathbf{u}_2(t)$. (This is tricky: the exponential in the formula for $\mathbf{u}_1$ might make you think the trajectory is curved. However, if you look carefully at the formula you will see $\mathbf{u}_1(t)$ is always a multiple of the vector $(1, 2)^T$.)

The applet *Linear phase portrait: matrix entries* will allow us to visualise this nicely, and get a feel for other sorts of trajectories. Later in this session you will look at this applet.