Structural Stability for Non-linear Systems

In the preceding note we discussed the structural stability of a linear system. How does it apply to non-linear systems?

Suppose our non-linear system has a critical point at $P$, and we want to study its trajectories near $P$ by linearizing the system at $P$.

This linearization is only an approximation to the original system, so if it turns out to be a borderline case, i.e., one sensitive to the exact value of the coefficients, the trajectories near $P$ of the original system can look like any of the types obtainable by slightly changing the coefficients of the linearization.

It could also look like a combination of types. For instance, if the linearized system had a critical line (i.e., one eigenvalue zero), the original system could have a sink node on one half of the critical line, and an unstable saddle on the other half. (This actually occurs.)

In other words, the method of linearization to analyze a non-linear system near a critical point doesn’t fail entirely, but we don’t end up with a definite picture of the non-linear system near $P$; we only get a list of possibilities. In general one has to rely on computation or more powerful analytic tools to get a clearer answer. The first thing to try is a computer picture of the non-linear system, which often will give the answer.

**Example.** $x' = y - x^2, \quad y' = -x + y^2$

Jacobian: $J(x, y) = \begin{pmatrix} -2x & 1 \\ -1 & 2y \end{pmatrix}$

Critical points: $y - x^2 = 0 \Rightarrow y = x^2$
$-x + y^2 = 0 \Rightarrow -x + x^4 = 0 \Rightarrow x = 0, 1.$
$\Rightarrow (0, 0)$ and $(1, 1)$ are the critical points.

$J(1, 1) = \begin{pmatrix} -2 & 1 \\ -1 & 2 \end{pmatrix}$:

characteristic equation: $\lambda^2 - 3 = 0 \Rightarrow \lambda = \pm \sqrt{3} \Rightarrow$ linearized system has a saddle.

This is **structurally stable** $\Rightarrow$ the nonlinear system has a saddle at $(1, 1)$.

$J(0, 0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$: eigenvalues $= \pm i \Rightarrow$ a linearized center.

This is **not structurally stable**. The nonlinear system could be any one of a
center, spiral out or spiral in. Using a computer program it appears that 
(0,0) is in fact a center. (This can be proven using more advanced methods.)

We can show the trajectories near (0,0) are not spirals by exploiting the 
symmetry of the picture. First note, if \((x(t), y(t))\) is a solution then so is 
\((y(-t), x(-t))\). That is, the trajectory is symmetric in the line \(x = y\). This 
implies it can’t be a spiral. Since the only other choice choice is that the 
critical point (0,0) is a center, the trajectories must be closed.

The following two examples show that a linearized center might also 
be a spiral in or a spiral out in the nonlinear system.

Example a. \(x' = y, y' = -x - y^3\).

Example b. \(x' = y, y' = -x + y^3\).

In both examples the only critical point is \((0,0)\).

\[J(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Rightarrow \text{linearized center. This is not structurally stable.}\]

In example a the critical point turns out to be a spiral sink. In example 
b it is a spiral source.

Below are computer-generated pictures. Because the \(y^3\) term causes the 
spiral to have a lot of turns we ‘improved’ the pictures by using the power 
1.1 instead.
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