Existence and Uniqueness and Superposition in the General Case

We can extend the results above to the inhomogeneous case.

\[ x' = A(t)x \text{ (homogeneous)} \]  
\[ x' = A(t)x + F(t) \text{ (inhomogeneous)}, \]

where \( F(t) \) is the input to the system.

**Linearity/superposition:**

1. If \( x_1 \) and \( x_2 \) are solutions to (H) then so is \( x = c_1x_1 + c_2x_2 \)
2. If \( x_h \) is a solution to (H) and \( x_p \) is a solution to (I) then \( x = x_h + x_p \) is also a solution to (I).
3. If \( x_1' = Ax_1 + F_1 \) and \( x_2' = Ax_2 + F_2 \) then \( x = x_1 + x_2 \) satisfies \( x' = Ax + F_1 + F_2 \). That is, superposition of inputs leads to superposition of outputs.

**proof:**

1. \( x' = c_1x_1' + c_2x_2' = c_1Ax_1 + c_2Ax_2 = A(c_1x_1 + c_2x_2) = Ax. \)
2. \( x' = x_h' + x_p' = Ax_h + Ax_p + F = A(x_h + x_p) + F = Ax + F. \)
3. \( x' = x_1' + x_2' = Ax_1 + F_1 + Ax_2 + F_2 = A(x_1 + x_2) + F_1 + F_2 = Ax + F_1 + F_2. \)

**Existence and uniqueness:** We start with an initial time \( t_0 \) and the initial value problem:

\[ x' = A(t)x + F(t), \ x(t_0) = x_0. \]  

**Theorem:** If \( A(t) \) and \( F(t) \) are continuous then there exists a unique solution to (IVP).