Problem 1: Find the critical points of the non-linear autonomous system

\[ \begin{align*}
    x' &= 1 - x + y \\
    y' &= y + 2x^2
\end{align*} \]

Solution: Critical points occur where \(1 - x + y = 0\) and \(y + 2x^2 = 0\). Substituting the first equation rewritten as \(y = x - 1\) into the second \(y + 2x^2 = 0\) we get

\[ 0 = x - 1 + 2x^2 \Rightarrow x = \frac{1}{2} \text{ or } x = -1. \]

Then \(x = \frac{1}{2} \Rightarrow y = -\frac{1}{2}\), and \(x = -1 \Rightarrow y = -2\).

Thus, the critical points are \(\left(\frac{1}{2}, -\frac{1}{2}\right)\) and \((-1, -2)\).

Problem 2: Write as equivalent first-order system and find the critical points:

\[ x'' - x' + 1 - x^2 = 0 \]

Solution: Let \(y = x'\), then \(y' = x'' = x' - 1 + x^2\). So the equivalent 2 \(\times\) 2 autonomous system is then

\[ \begin{align*}
    x' &= y \\
    y' &= y - 1 + x^2
\end{align*} \]

Critical points occur when \(y = 0\) and \(y - 1 + x^2 = 0 \Rightarrow y = 0\) and \(x^2 = 1\).

So, the critical points are \((1, 0)\) and \((-1, 0)\).

Problem 3: In general, what can you say about the relation between the trajectories and the critical points of the system on the left below, and those of the two systems on the right?

\[ \begin{align*}
    x' &= f(x, y) \\
    y' &= g(x, y) \\
    a) x' &= -f(x, y) \\
    y' &= -g(x, y) \\
    b) x' &= g(x, y) \\
    y' &= -f(x, y)
\end{align*} \]
Solution: (a) For this system the tangent vector \((-f(x, y), -g(x, y))\) to a trajectory is equal in magnitude but opposite in direction to the tangent vector \((f(x, y), g(x, y))\) to the original system. So the trajectory paths are the same but are traversed in the opposite direction.

The left hand figure is the original system \(x' = .6x + y, \; y' = -x\).
The right hand figure is the reversed system \(x' = -.6x - y, \; y' = x\).

The critical points are the same for both systems, they occur at \(f(x, y) = 0\) and \(g(x, y) = 0\).

(b) For this system the tangent vector \((g(x, y), -f(x, y))\) to a trajectory is perpendicular to the tangent vector \((f(x, y), g(x, y))\) to the original system. So the solutions to (b) are the orthogonal trajectories to the original system.

The left hand figure is the original system \(x' = .6x + y, \; y' = -1.5x\).
The right hand figure is the orthogonal system \(x' = -1.5y, \; y' = .6x + y\).

The critical points occur at \(g(x, y) = 0\) and \(-f(x, y) = 0\), i.e. they are the same as for the original system.