Part I Problems and Solutions

Problem 1: Give the general solution to the DE system $x' = \begin{bmatrix} -2 & 1 \\ -1 & -4 \end{bmatrix} x$ and also give its phase-plane picture (i.e. its direction field graph together with a few typical solution curves).

Solution: Characteristic equation $\lambda^2 + 6\lambda + 9 = (\lambda + 3)^2 = 0 \rightarrow$ repeated root $\lambda = -3$.

The single eigenvector $v$ and a generalized eigenvector $w$ such that $(A - \lambda I)w = v$, and the scalar component functions $x_1(t), x_2(t)$ of the general solution $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ of the form $x(t) = c_1v e^{\lambda t} + c_2(v t + w) e^{\lambda t}$ of the given system $x' = Ax$ are as follows:

Eigenvector: $v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Generalized eigenvector: $w = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Thus, $x_1(t) = (c_1 + c_2 + c_2 t)e^{-3t}$ and $x_2(t) = (-c_1 - c_2 t)e^{-3t}$.

Problem 2: For each of the following linear systems, carry out the graphing program laid out in this session, that is:

(i) find the eigenvalues of the associated matrix and from this determine the geometric type of the critical point at the origin, and its stability;
(ii) if the eigenvalues are real, find the associated eigenvectors and sketch the corresponding trajectories, showing the direction of motion for increasing $t$; then draw some nearby trajectories;

(iii) if the eigenvalues are complex, obtain the direction of motion and the approximate shape of the spiral by sketching in a few vectors from the vector field defined by the system.

a) $x' = 2x - 3y, y' = x - 2y$

b) $x' = 2x, y' = 3x + y$

c) $x' = -2x - 2y, y' = -x - 3y$

d) $x' = x - 2y, y' = x + y$

e) $x' = x + y, y' = -2x - y$

Solution: Let $\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ throughout, and $M$ be such that $\vec{x}'(t) = M\vec{x}(t)$. Let $M$ have eigenvalues $\lambda_1, \lambda_2$, with corresponding eigenvectors $\vec{v}_1, \vec{v}_2$. The general solution is thus

$$\vec{x}(t) = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2 e^{\lambda_2 t}$$

a) $M = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$, with eigenvalues $\pm 1$ and eigenvectors $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The system has a critical point at $(0, 0)$ which is a saddle point.

For $c_1 = 0$ and as $t \to \infty$, $\vec{x}(t) = c_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Similarly, for $c_2 = 0$ and $t \to -\infty$, $\vec{x}(t) \to \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Thus the behavior near the saddle point looks like
b) \( M = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} \), with eigenvalues 2, 1 and eigenvectors \( \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \). The system has an unstable node at (0,0).

As \( t \to -\infty \), all trajectories go to \( \vec{0} \).

Thus the behavior near the node looks like

\[
\begin{align*}
\text{For } t \to -\infty, & \quad c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t \text{ is dominant term, so the solutions are near the } y\text{-axis. For } t \to \infty, \\
c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{2t} \text{ dominates, so solutions are parallel to } \begin{bmatrix} 1 \\ 3 \end{bmatrix}.
\end{align*}
\]

c) \( M = \begin{bmatrix} -2 & -3 \\ -1 & -3 \end{bmatrix} \), with eigenvalues \(-4, -1\) and eigenvectors \( \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \). The system has an asymptotically unstable node at (0,0). As \( t \to \infty \), all trajectories go to \( \vec{0} \). The behavior near the origin looks like:
For $t \to -\infty$, $\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-4t}$ dominates, so solutions are parallel to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$; for $t \to \infty$, $\begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{-t}$ dominates, so solutions come in to the origin asymptotic to the line with direction vector $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

d) $M = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$, eigenvalues $1 \pm i\sqrt{2}$. The system then has an unstable spiral around (0,0).

Near $y = 0$, $x' \approx x$, so $x$ is increasing where the spiral cuts the positive $x$-axis. As $y$ increases, so does $e^t$, so the spiral is outwards from the origin.

e) $M = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$. Eigenvalues are $\pm 1$, pure imaginary, so the system has a stable center.

The curves are ellipses, since $\frac{dy}{dx} = \frac{-2x - y}{x + y}$ which integrates easily after cross-multiplying.
to \(2x^2 + 2xy + y^2 = c\).

Direction of motion: For instance, at \((1,0)\) the vector field is \(x' = 1, y' = -2\), so motion is clockwise.