18.04 Final Exam from Fall 2002

Wednesday, December 18, 2002  Time: 9 am - 12 noon

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Final Examination in
18.04 Complex Variables with Applications

NOTE: Students are not allowed to use any books or notes during this examination. If brought into the room, such items must not be left nearby.

Personal calculators may be used, but not shared.

1. Use complex arithmetic (rather than your trusty calculator) to demonstrate anew that
   \[ z = 2 \cos 36^\circ = e^{i\pi/5} + e^{-i\pi/5} \]
   is one solution of \( z^4 - 3z^2 + 1 = 0 \).

2. Where can we possibly be if
   (a) \( z^6 + z^4 + z^2 + 1 = 0 \)
   or if (b) \( \sin z = \cos z \) ?

3. Somehow or other, show again that
   \[ \frac{d}{dz} \tan^{-1} z = \frac{1}{1 + z^2} \]
   even for complex \( z \)’s.

4. Evaluate \( \oint_{|z|=1} z^5 e^{2/z} \, dz \) once around that unit circle.

5. Let \( C \) be the ellipse \( x^2/4 + y^2/9 = 1 \) traversed once in the counterclockwise direction, and define
   \[ G(z) = \oint_C \frac{z^2 - z + 2}{z - z} \, dz \]
   for any \( z \) inside \( C \).
   Then find \( G(i) \), \( G'(i) \) and \( G''(-1) \).

6. Evaluate \( \int_{-\infty}^{\infty} \frac{x}{\sinh x} \, dx \),
   where \( \sinh x = (e^x - e^{-x})/2 \) as usual.

7. (a) Which region of the complex \( z \)-plane gets mapped by
   \[ w(z) = \frac{z - 1}{z + 1} \]
   into the interior of the circle \( |w| = 1 \), and why?
   (b) Use the above answer as a clue to find a related Möbius (or bilinear) transformation \( W(z) \) that carries the top half of the \( z \)-plane into a unit circle centered instead at \( W = 1 + i \).

8. Find that Fourier series of period \( 2\pi \) which represents the function
   \[ f(\theta; a) = \frac{1}{1 + a \cos \theta} \]
   for any real \( a \), subject only to the requirement that \( |a| < 1 \).
   HINT: Consider terms like \( p|k| e^{ik\theta} \) in a complex FS.
1. After squaring, etc. \( z^4 - 3z^2 + 1 = e^{i4\pi/5} + e^{i2\pi/5} + 1 + e^{-i2\pi/5} + e^{-i4\pi/5} = 0 \)

2. (a) \( z^2 = \text{either } i \text{ or } -1 \text{ or } -i \). Hence \( z = e^{i\pi/4} \text{ or } e^{i\pi/2} \text{ or } e^{i3\pi/2} \)

(b) Need \( e^{iz} - e^{-iz} = i(e^{iz} + e^{-iz}) \) or \( e^{2iz} = i \rightarrow \text{ only } z = \frac{\pi}{4} \pm \frac{\pi}{2} \text{ all real} \)

3. Well, if \( w = \tan^{-1} z \), then \( z = \tan w \). Hence \( \frac{dz}{dw} = 1 = \frac{1}{\cos^2 w} \text{... } \frac{dw}{dz} = \cos^2 w \text{... } \frac{1}{\cos^2 w} \text{... } \frac{d^2 w}{dz^2} = 1/(1+z^2) \text{... QED.} \)

4. Since \( e^{2iz} \) has laurent series \( 1 + \frac{2}{2!}z^2 + \frac{4}{3!}z^4 + \frac{8}{4!}z^6 + \frac{16}{5!}z^8 + \frac{32}{6!}z^{10} + \frac{64}{7!}z^{12} + \ldots \), the residue at \( z = 0 \) for \( z^5 e^{2iz} \) equals \( 64/720 \ldots \) or our \( \frac{8\pi}{45} \)

5. Writing \( f(z) = z^2 - z + 2 \), we have from the C.I.F. (incl. derivative version) that \( G(\alpha) = 2\pi i f(\alpha) = \frac{4\pi i}{4!} i \), \( G'(\alpha) = 2\pi i f'(\alpha) = -2\pi (2+i) \),

and \( G''(\alpha) = 2\pi i f''(\alpha) = \frac{2\pi i}{2!} \frac{1}{2!} = \frac{i}{2} \).

6. From solns to our Exam 3, that's \( \int_0^\infty \frac{x}{\sinh x} \, dx = \frac{\pi^2}{2} \) again.

7. (a) Answer: right half-plane, or \( Re z > 0 \ldots \) because for all those \( z \)'s, but no others — our numerator \( |z-1| < |\text{denominator}| = |z+1| \)

(b) Clearly \( p(z) = \frac{z-i}{z+i} \) will do to the upper half-plane what \( W(z) \) did to the right HP. (And \( q(z) = p(z) e^{i\theta} \) will "spin" the answer plane ... but why complicate, since \( \delta = 0 \) will suffice.)

Still need to add \( 1+i \) to shift the center \( \rightarrow W(z) = \frac{z-i}{z+i} e^{i\theta} + 1+i \)

8. Taking the HINT, consider \( S(\theta) = \ldots + pe^{-i\theta} + 1 + pe^{-i\theta} + p^2 e^{-2i\theta} + p^3 e^{-3i\theta} + \ldots \)

or \( S(\theta) = \frac{1}{1-pe^{-i\theta}} + \frac{1}{1-pe^{-i\theta}} - 1 \equiv \frac{1-p^2}{1+p^2 - 2p \cos \theta} \).

So we have a match provided \( \frac{\theta}{1+p^2} = \alpha \), although the desired FS will be \( \frac{1+p^2}{1-p^2} \) times the sum \( S(\theta) \) given above.
Friday, May 21, 1999  

MASSACHUSETTS INSTITUTE OF TECHNOLOGY  
Final Examination in  
18.04 Complex Variables with Applications

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1. Use complex algebra to confirm that

(a) the sum $\alpha + \beta + \gamma$ of the three angles shown is $\pi/2$

(b) the real part of any solution of $(z+1)^8 = (z-1)^8$ must be zero

2. For any real $x$ and $N = 0, 1, 2, 3, \ldots$, evaluate the sum

$$S_N(x) = \sum_{k=-N}^{N} e^{ikx}$$

as the ratio of two sines, and verify afterwards that this formula really works for both $N = 0$ and $N = 1$.

3. The inventor of the brilliant new function

$$w = \text{itch}(z) = 2z^2 + e^{2z}$$

wants to know its inverse $\text{itch}^{-1}(w)$ explicitly in terms of the complex logarithm. Please help ... and also show the power of your formula (or of your logic) by reporting all possible values of $z$ for which $\text{itch}(z) = 3$.

4. Evaluate $\int_{|z|=1} z^4 e^{1/z} \, dz$. HINT: Think of Laurent

5. Evaluate $\int_{0}^{\infty} \frac{dx}{x^6 + 1}$. HINT: A slice of pizza

6. Show that the function $w(z) = z + \frac{1}{z}$ indeed maps any circle $|z| = \text{const}$ from the $z$-plane into an ellipse in the $w$-plane, with foci at $w = \pm 2$. (Sure, you may have forgotten exactly how the focus of an ellipse was defined ... but if need be, as here is, please rederive that as well.)

7. Solve the Laplace equation $\nabla^2 T = 0$ OUTSIDE the unit circle, given that

$$T(r=1, \theta) = \cos^6 \theta$$

and also that $T(r, \theta)$ approaches the constant value 5/16 at all large radii $r$. HINT: Remember $z^{-n}$

8. Find the first three coefficients $a_0$, $a_2$ and $a_4$ needed in this Fourier cosine series of period $\pi$:

$$|\sin x| = a_0 + a_2 \cos 2x + a_4 \cos 4x + \ldots$$

PS: Since most of you are about to leave town, let's skip the usual funny business with code names for grade-postings outside an office door. Instead, a discreet email from you after this coming weekend will probably disclose your final grade very swiftly and reliably, even from afar! Now have a good summer ...
1. a) Consider \((3+i)(2+i)(1+i) = (5+5i)(1+i) = 10i\) with \(\theta = 90^\circ\)
   
b) Here \(|z+d| = |z-d|\), or \((x+d)^2 + y^2 = (x-d)^2 + y^2 \rightarrow 4x = 0\)

2. Here \(S_n(x) = \sin(n+\frac{1}{2})x / \sin(\frac{1}{2}x)\), as explained most recently as the soln.
   to Prob 3 from Exam #1.

   The special case \(N=0\) yields \(S_0(x) = 0\) most reasonably, whereas for \(N=1\) we expect
   \(S_1(x) = e^{ix} + d + e^{-ix} = 2 \cos x\) and get \(\lim_{N \to \infty} \sin(x+\frac{\pi}{2}) = \sin x \cos \frac{\pi}{2} + \cos x \sin \frac{\pi}{2} = 2 \sin \frac{\pi}{2} \cos \frac{\pi}{2} + \cos x \sin \frac{\pi}{2} = \sin \frac{\pi}{2} x \cos x + \cos x \sin \frac{\pi}{2} \rightarrow S_1(x) = \sin x\)

3. Here \(z = \frac{\cos \sqrt{w^2 - d^2} - d}{w}\) and \(w=3\) \(\rightarrow z = \frac{2}{6} = \frac{1}{3}\) for more detail, see soln. to Prob 3 from Exam #1

4. Since \(e^{4\pi i} = 1 + \frac{4}{1} + \frac{4}{2!} + \frac{4}{3!} + \ldots\), the residue \(Res(0)\) of \(e^{4\pi i}x^4\) is the
   coefficient \(\frac{4}{5!}\) of the resulting \(\frac{4}{x}\) term. Hence \(\int e^{4\pi i}x^4 dx = 2\pi i / 5!\)

5. The integral \(I = \int_0^\infty \frac{e^{m/n} \, dt}{x^2 + 1}\), as can be inferred from
   
   \[\int_0^\infty e^{m/n} \, dt = \frac{\pi}{3}\]

   and \(J = \int_0^\infty \frac{\frac{e^{m/n} \, dt}{x^2 + 1}}{x^2 + 1}\) and the residue \(Res(e^{m/n}) = \frac{e^{m/n}}{e^n}\).

6. The circle \(z = Re + i \text{Im} \theta\) translates into
   \(w = (R e^{i \theta}) \cos \theta + i (R e^{i \theta}) \sin \theta \equiv u + iv = \text{Re} + i \text{Im} \theta\)
   Hence indeed the squared focal distance \(\sqrt{a^2 - b^2} = (R e^{i \theta})^2 + (R e^{i \theta})^2 = 4\)

7. Since \(\cos^2 \theta = \frac{1}{2} (e^{i \theta} + e^{-i \theta})^6 = \frac{1}{32} \cos 6\theta + \frac{6}{32} \cos 4\theta + \frac{15}{32} \cos 2\theta + \frac{20}{64} \cos 0\theta\),
   our desired exterior solution
   \(T(r, \theta) = \frac{5}{16} \cos 6\theta + \frac{15}{32} \cos 4\theta + \frac{30}{32} \cos 2\theta + \frac{20}{64} \cos 0\theta\)

8. Here \(|\sin x| = \frac{2}{\pi} - \frac{4}{5\pi} \cos 2x - \frac{4}{15\pi} \cos 4x - \ldots\) to the extent that we requested

   ... although in fact this also equals \(\frac{2}{\pi} \{1 - \frac{2}{3} \cos 2x - \frac{2}{5} \cos 4x - \frac{2}{7} \cos 6x \ldots\}\)
   from which, since \(\frac{2}{3} = \frac{1}{3} - \frac{1}{3} \), \(\frac{2}{5} = \frac{1}{3} - \frac{2}{7} \), \(\frac{2}{9} = \frac{1}{3} - \frac{4}{15}\), etc., it is \text{FUN}
   to sum this at \(x=0\) to obtain \(\frac{2}{\pi} \{1 - \frac{1}{3} \} = 0\) very convincingly. And
   even at \(x = \frac{\pi}{3}\) this checks nicely, since \(1 - \frac{1}{3} + \frac{1}{3} - \frac{4}{15} + \ldots = \frac{2}{\pi} = \text{Wallis formula}\)

   Now have a \text{NICE SUMMER!}
Thursday, May 22, 1997

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1. If \(|z| = 1\), prove that:
   a) \(|z - w| = |1 - \bar{w}z|\) for any \(w\)
   b) \(\text{Re}[1/(1-z)] = 1/2\), excluding only \(z = 1\)

2. For the function \(f(z) = \frac{\sin z \cos 3z}{z^4}\):
   a) Find the first three non-zero terms of its innermost Laurent expansion about \(z = 0\)
   b) Integrate \(\oint f(z) \, dz\) counterclockwise once around \(|z| = 1\)

3. Use residue calculus to evaluate:
   a) \(I = \int_{-\infty}^{\infty} \frac{dx}{x^2 + x + 1}\)
   b) \(J = \int_{-\infty}^{\infty} \frac{dx}{x^4 + x^2 + 1}\)

4. Find the steady-state or "harmonic" temperature at the center of a unit disk if the temperature at its rim is:
   \[T(1,\theta) = \cos^4 \theta\]
   \[T(1,\theta) = 3 / (2 - \cos \theta)\]

5. Figure out that Fourier series of period \(\pi\) (and also \(2\pi\)) which closely imitates the non-negative function \(f(x) = |\cos x|\)

6. Regarding Möbius transformations like \(w = \frac{az + b}{cz + d}\):
   a) Show that such transformations can never have more than two fixed points (at which \(w(z) = z\)) unless they are just trivial "carbon copies" with \(w = z\) everywhere
   b) Locate both fixed points for \(w = (z-1) / (z+1)\)
   c) Find a different \(w = (az+b) / (cz+d)\) that maps the points \(z = 0, 1, \infty\) into \(w = 0, 1, 2\), respectively
4. a) \(|z - w| = |\bar{z} - \bar{w}| = \left| \frac{z}{\bar{z}} - \frac{w}{\bar{w}} \right| = \left| \frac{z - w}{\bar{z} - \bar{w}} \right| = |d - \bar{w}z| \quad \checkmark
\]
    \[\text{since } z \bar{z} = 1\text{ here} \quad \text{Choose } z = 1\]

b) Writing \(z = e^{i\theta}\):
\[
\frac{d}{dz} = \frac{d}{d\theta} \quad \frac{d}{d\theta} = \frac{(d - \cos \theta) + i \sin \theta}{(d - \cos \theta) - i \sin \theta} = \frac{(d - \cos \theta) + i \sin \theta}{(d - \cos \theta) - i \sin \theta} = \frac{e^{-i\theta}}{e^{i\theta}} = e^{-2i\theta} \quad \checkmark \text{since } \theta = 0.
\]  

2. a) \(f(z) = \frac{d}{dz} \left( z - \frac{z^3}{3} - \frac{z^5}{10} - \cdots \right) \quad \text{Residue } = \frac{1}{3} \]
\[
\frac{d}{dz} - \frac{z^3}{3} - \frac{z^5}{10} \quad \text{b) } \text{Residue} = \frac{-2\pi i}{3}.
\]

3. a) \(I = \frac{2\pi}{\sqrt{3}} \quad \text{...from simple pole at } z = e^{i\pi/6}, \quad \text{Residue} = \frac{4}{i\sqrt{3}}\)
\[b) \quad J = \frac{\pi}{\sqrt{3}} \quad \text{...from simple pole at } z = e^{i\pi/3}, e^{-2i\pi/3} \quad \text{Residue} = \frac{-i\sqrt{3}}{6}.
\]

4. a) \(T(\theta) = 3 \quad \text{...literally the mean of these rim values}\)
\[b) \quad T(\theta) = \frac{\cos^2 \theta}{2} \quad \text{a) } \cdots = \frac{3}{2\pi} \int_0^{2\pi} \frac{d\theta}{2 - \cos \theta} = \frac{\sqrt{3}}{3}.
\]

5. \(f(x) = |\cos x| = a_0 + a_2 \cos 2x + a_4 \cos 4x + \cdots \quad \text{yet, only the even } a_k \text{ survives, by symmetry}\)
\[\text{and also } \quad a_0 = \left< |\cos x| \right> = \frac{2}{\pi}.
\]

The others are
\[(\text{for } k=1,3,5,\ldots) \quad a_{2k} = \frac{1}{\pi} \int_{-\pi}^{\pi} |\cos x| \cos 2kx \, dx = \frac{(-1)^{k+1} \pi}{4k^2} \]
\[
= \frac{\pi}{4} \int_{-\pi/2}^{\pi/2} e^{i2kx} \left( e^{ix} - e^{-ix} \right) \frac{1}{2} \, dx = \frac{\pi}{4} \left[ e^{i2kx} \left( \frac{e^{i2kx} - e^{-i2kx}}{2i} \right) \int_{-\pi/2}^{\pi/2} \right] \]

6. a) If \(\frac{a_2 + b}{c_2 + d} = \bar{z}_0\), then of course \(a_2^2 + (d-a) \bar{z}_0 - b = 0\)
\[\text{with 2 roots at most, unless } c=0 \text{ and } d \neq a \text{ and } b = 0 \text{ too.}\]

b) For \(\frac{\bar{z}_0 - i}{\bar{z}_0 + i} = \bar{z}_0 \) in particular, \[\bar{z}_0 = \frac{d + \sqrt{d^2 - 4}}{2} (d-i)\]
\[w = \frac{\bar{z}_0}{d + \sqrt{d^2 - 4}} \text{ will do fine.}\]

\[\text{PS: Only if you REALLY need to know your final grade in a hurry, send me an email by Friday morning.}\]
\[\text{I leave on a weeklong trip Friday afternoon. Good summer, etc.}\]
\[\text{AT}\]