18.04 Practice problems for final exam, Spring 2018 Solutions

On the final exam you will be given a copy of the Laplace table posted with these problems.

Problem 1.

Which of the following are meromorphic in the whole plane.

(a) $z^3$
(b) $z^{5/2}$
(c) $e^{1/z}$
(d) $1/\sin(z)$.

answers: Meromorphic means analytic except for poles of finite order.
(a) Yes, this is entire.
(b) No, this requires a branch cut in the plane to define a region where it’s analytic.
(c) No, the singularity at $z = 0$ is an essential singularity, not a finite pole.
(d) Yes, $\sin(z)$ has simple zeros at $n\pi$ for all integers $n$. So $1/\sin(z)$ has simple poles at these points.

Problem 2.

(a) Let $f(z) = \frac{(z - 2)^2z^3}{(z + 5)^3(z + 1)^3(z - 1)^4}$. Compute $\int_{|z|=3} \frac{f'(z)}{f(z)} \, dz$

(b) Find the number of roots of $g(z) = 6z^4 + z^3 - 2z^2 + z - 1 = 0$ in the unit disk.

(c) Suppose $f(z)$ is analytic on and inside the unit circle. Suppose also that $|f(z)| < 1$ for $|z| = 1$. Show that $f(z)$ has exactly one fixed point $f(z_0) = z_0$ inside the unit circle.

(d) True or false: Suppose $f(z)$ is analytic on and inside a simple closed curve $\gamma$. If $f$ has $n$ zeros inside $\gamma$ then $f'(z)$ has $n-1$ zeros inside $\gamma$.

answers: (a) By the argument principle the $\int_{\gamma} \frac{f'}{f} \, dz = 2\pi i (Z_{f,\gamma} - P_{f,\gamma})$. In this case, the zeros of $f$ inside $\gamma$ are 2, 0 of order 2 and 3 respectively. The poles inside $\gamma$ are $-1$ and $1$ of order 3 and 4 respectively. So, the integral equals

$$2\pi i(2 + 3 - 3 - 4) = -4\pi i.$$ 

(b) On the unit circle $|z^3 - 2z^2 + z - 1| < 5$ and $|6z^4| = 6$. Therefore by Rouche’s theorem the number of zeros of $g(z)$ inside the unit circle is equal to the number of zeros of $6z^4$, i.e. 4.

(c) Let $g(z) = f(z) - z$. We want to show $g$ has exactly one root inside the unit circle. We know $|f(z)| < |z - z| = 1$ on the unit circle. So by Rouche’s theorem $g(z)$ and $-z$ have the same number of zeros in the unit disk. That is, they both have exactly one such zero. QED.

(d) False. Consider $f(z) = e^z - 1$. This has 3 zeros inside the circle $|z| = 3\pi (0, \pm 2\pi)$. But $f'(z) = e^z$ has no zeros.

Problem 3.

Let $A = \{ z \mid 0 \leq \text{Re}(z) \leq \pi/2, \, \text{Im}(z) \geq 0 \}$.
Let $B =$ the first quadrant.

Show that $f(z) = \sin(z)$ maps $A$ conformally onto $B$

**answers:** (a) You should supply a picture of the regions $A$ and $B$ and develop a picture tracking the argument we give. We see where $f$ maps the boundary of $A$. The boundary of $A$ has 3 pieces:

Piece 1: $z = iy$, with $y \geq 0$. On this piece

\[
\sin(z) = \frac{e^{-y} - e^{y}}{2i} = \frac{(e^{y} - e^{-y})}{2}i
\]

So, the image of piece 1 is the positive imaginary axis.

Piece 2: $z = x$, with $0 \leq x \leq \pi/2$. On this piece $\sin(z) = \sin(x)$, so the image runs from 0 to 1 along the real axis.

Piece 3: $z = \pi/2 + iy$, with $y \geq 0$. On this piece

\[
\sin(z) = \frac{e^{-y+\pi i/2} - e^{-\pi i/2}}{2i} = \frac{(ie^{-y} + ie^{-y})}{2i} = \frac{2}{2} = \cosh(y).
\]

So, the image of piece 3 is the real axis greater than 1.

We have shown that $f(z)$ maps the boundary of $A$ to the boundary of $B$.

To see that $A$ is mapped to $B$ it’s enough to verify that one point inside $A$ is mapped to a point inside $B$. There are lots of ways to do this. Here’s one. We know

\[
\sin(x + iy) = \frac{e^{-y+ix} - e^{y-ix}}{2i}.
\]

Pick $x = \pi/4$ and $y$ so large that $e^{-y}$ is very tiny. Then

\[
\sin(x + iy) \approx -e^{y}e^{-ix}2i = -e^{y}\sqrt{2}/2 - i\sqrt{2}/2 = e^{y}\sqrt{2} + i\sqrt{2}
\]

This last value is clearly in the first quadrant, i.e inside $B$. 
