18.04 Recitation 11  
Vishesh Jain

1.1. Find an LFT from the half-plane $H_a := \{(x, y) : y > x \tan(\alpha)\}$ to the unit disc $D_1$ centered at the origin.

**Ans:** First, rotate by $\alpha$ clockwise to map $H_a$ to the upper half-plane $H$. Then, use $T(z) = \frac{z-i}{z+i}$ to the map $H$ to the unit-disc.

1.2. Find a conformal map from the strip $I_x := \{(x, y) : 0 < y < \pi\}$ to the upper half-plane $H$.

**Ans:** $e^z$.

1.3. Find a conformal map from the upper semi-disc $R_2 := \{(x, y) \in D_1 : y > 0\}$ to the upper half-plane $H$.

**Ans:** First, map $z_1 = 1$, $z_2 = i$ and $z_3 = -1$ to the three points $w_1 = 0$, $w_2 = 1$ and $w_3 = \infty$. This can be accomplished by the LFT

$$T_2(z) = \frac{z - z_1}{z - z_3} \frac{z_2 - z_3}{z_2 - z_1} = \frac{(z - 1)(i + 1)}{(z + 1)(i - 1)} = -i \frac{z - 1}{z + 1}.$$

Since $T_2(0) = i$, it follows that the segment $(-1, 1)$ on the real axis is mapped to the positive imaginary axis. It follows that $T_2$ maps the upper semi-disc to the first quadrant $Q_1 = \{(x, y) : x > 0, y > 0\}$. Now, use $z^2$ to map $Q_1$ to $H$.

1.4. Find a conformal map from the “infinite well” $W_x := \{(x, y) : 0 < y < \pi, x < 0\}$ to the upper half-plane.

**Ans:** First, use $e^z$ to map $W_x$ to the upper semi-disc $R_2$. Next, use the map from 2.3. to map $R_2$ to the upper half-plane.

2.1 Find the reflection of a point $z_1$ in the $x$-axis.

**Ans:** $\overline{z_1}$.

2.2. Define the reflection $r_C(z_2)$ of a point $z_2$ in a circle $C$ as follows. Let $T_{CL}$ be an LFT mapping the circle $C$ to a line $L$. Then, $r_C(z_2) := T_{CL}^{-1}(r_L(T_{CL}(z_2)))$, where $r_L$ denotes reflection in the line $L$. Use this definition to find the reflection of a point $z_2$ in the unit circle.

**Ans:** We know that the LFT

$$T^{-1}(z) = \frac{z - i}{z + i}$$

maps the $x$-axis to the unit circle. Therefore,

$$T(z) = \frac{z + 1}{-z + 1}$$
maps the unit circle to the x-axis. Computing the above expression directly now gives that the reflection of $z_2$ in the unit circle is $\frac{1}{z_2}$, which is what we expect.
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