1. Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function. We write $f(x, y) = u(x, y) + iv(x, y)$. Suppose that $u$ and $v$ are $C^2$ i.e. all partial derivatives of $u$ and $v$ of order up to (and including) 2 exist, and are continuous. Show that $f' = \frac{df}{dz} : \mathbb{C} \to \mathbb{C}$ is also analytic.

**Ans:** $f' = u_x + iv_x$. Let $U = u_x, V = v_x$. Then, $U_x = u_{xx}, U_y = u_{xy}, V_x = v_{xx}, V_y = v_{xy}$. Therefore, $U_x = u_{xx} = (v_y)_x = v_y = v_{xx} = V_y$. Also, $U_y = u_{xy} = -v_{xx} = -V_x$.

2.1. Show that $\int \bar{z} dz$ is not path independent in $\mathbb{C}$. Why does this not contradict the fundamental theorem for complex line integrals?

**Ans:** Consider $\int_\gamma \bar{z} dz$ where $\gamma$ is the unit circle centered at the origin. Parameterizing $\gamma$ by $\gamma(t) = e^{it}$, we get $\int_\gamma \bar{z} dz = \int_0^{2\pi} e^{-it} ie^{it} dt = 2\pi i$.

2.2. For each $n \in \mathbb{Z}$, compute $\int_\gamma z^n dz$, where $\gamma$ is the unit circle centered at the origin. Are your answers consistent with the fundamental theorem?

**Ans:** For $n \geq 0$, this is 0 by the fundamental theorem since $z^n = \frac{1}{n+1} \frac{d}{dz} z^{n+1}$ on $\mathbb{C}$.

For $n < -1$, this is 0 by the fundamental theorem since $z^n = \frac{1}{n+1} \frac{d}{dz} z^{n+1}$ on $\mathbb{C}\setminus\{0\}$, and $\gamma$ is completely contained in $\mathbb{C}\setminus\{0\}$.

For $n = -1$, parameterize $\gamma$ by $\gamma(t) = e^{it}$ to get $\int_\gamma z^{-1} dz = \int_0^{2\pi} e^{-it} ie^{it} dt = 2\pi i$.

2.3. Do any of the answers in 2.2 change if $\gamma$ is a circle such that the disk bounded by the circle does not contain the origin?

**Ans:** All the answers are now zero, since we can enclose a circle not containing the origin in a region where $\log(z)$ is analytic, and then we can use that $z^{-1} = \frac{d}{dz} \log(z)$ in such a region.

3. Recall from Recitation 2 that $\cos(z) = \cos(x) \cosh(y) - i \sin(x) \sinh(y)$.

3.1. Consider the region $\mathcal{R} = \{(x, y) \in \mathbb{R}^2 : 0 < x < \pi\}$. What are the images of horizontal and vertical lines in $\mathcal{R}$? Is the mapping $z = \cos(z)$ restricted to $\mathcal{R}$ a one-to-one mapping?

**Ans:** Vertical lines $x_0 + it$ are sent to $\cos(x_0 + it) = \cos(x_0) \cosh(t) - i \sin(x_0) \sinh(t)$.

Viewed as a map to $\mathbb{R}^2$, this is $(\cos(x_0) \cosh(t), -\sin(x_0) \sinh(t))$. This satisfies

$$\frac{u^2}{\cos^2(x_0)} - \frac{v^2}{\sin^2(x_0)} = 1$$

which is the equation of a hyperbola.

Horizontal lines $t + iy_0$ are sent to $\cos(t + iy_0) = \cos(t) \cosh(y_0) - i \sin(t) \sinh(y_0)$.

This satisfies

$$\frac{u^2}{\cosh^2(y_0)} + \frac{v^2}{\sinh^2(y_0)} = 1$$
which is the equation of an ellipse.

See attached figure.
3.2. To $\mathcal{R}$, add the half lines $x = 0, y \geq 0$ and $x = \pi, y > 0$ to produce a new region $\mathcal{R}_1$. What is the image of $\mathcal{R}_1$ under the map $z \mapsto \cos(z)$? Is the map still one-to-one on $\mathcal{R}_1$?

**Ans:** The image consists of the entire complex plane. Yes, the map is one-to-one.

3.3. Note that $\mathcal{R}_1$ gives a branch of the multi-valued function $\cos^{-1}(z)$. What are the branch cuts in the domain of $\cos^{-1}(z)$ for this branch?

**Ans:** $(-\infty, -1) + i0$ and $(1, \infty) + i0$. Note that as you cross these segments vertically (say from up to down), $\cos^{-1}(z)$ jumps sharply.