18.04 Recitation 4
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1. We will compute $I = \int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2} \, dx$ using Cauchy’s integral formula. It will be helpful to recall the triangle inequality for integrals: $|\int_C f(z) \, dz| \leq \int_C |f(z)| \, |dz|$.

1.1. Consider the semicircle $C$ in the upper half plane which is centered at 0 and has radius $R$. Use Cauchy’s integral formula to compute $\int_C \frac{1}{(1+z^2)^2} \, dz$.

1.2. Decompose $C = C_1 \cup C_2$, where $C_1$ denotes the segment between $-R$ and $R$ on the $x$-axis, and $C_2$ denotes the remaining part of $C$. Use the triangle inequality for integrals to give an upper bound on $\left| \int_{C_2} \frac{1}{(1+z^2)^2} \, dz \right|$.

1.3. Use the results of the previous two parts to obtain an estimate $\int_{C_1} \frac{1}{(1+z^2)^2} \, dz$. What happens as you take $R \to \infty$?

**Ans:** See Example 4.11 in the notes.

2.1. (Cauchy’s Inequality) Let $C_R$ be the circle of radius $R$ centered at the point $z_0$, and suppose that $f$ is analytic on $C_R$ and its interior. Further, let $M_R = \max_{z \in C_R} |f(z)|$. Use Cauchy’s integral formula for derivatives, and the triangle inequality for integrals to show that

$$\left| f^{(n)}(z_0) \right| \leq \frac{n! M_R}{R^n}.$$

**Ans:** See Theorem 4.15 in the notes.

2.2. (Liouville’s Theorem) Now, suppose $f$ is an entire function and $|f(z)| \leq M$ for all $z \in \mathbb{C}$. By analyzing the $n = 1$ case in the previous part, what can you say about $f$?

**Ans:** See Theorem 4.16 in the notes.

3. (Fundamental Theorem of Algebra) Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ be a degree $n$ polynomial with $a_n \neq 0$. We will show that $P(z)$ has exactly $n$ roots (counting multiplicities) over $\mathbb{C}$.

3.1. Assume for contradiction that $P(z) \neq 0$ for all $z \in \mathbb{C}$. Show that under this assumption, $f(z) := 1/P(z)$ is entire and bounded. Use Liouville’s theorem to get a contradiction.

3.2. The previous part shows that $P$ must have at least one root. Iterate it to show that $P$ has exactly $n$ roots (counting multiplicities).

**Ans:** See Section 4.7.2. of the notes.

4. (Mean Value Property) Let $C_R$ be the circle of radius $R$ centered at the point $z_0$, and suppose that $f$ is analytic on $C_R$ and its interior. Use Cauchy’s integral formula to show that

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{i\theta}) \, d\theta.$$
Ans: See Theorem 4.18 in the notes.