18.04 Recitation 8
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1.1 Show that if \( g(z) \) has a simple zero at \( z_0 \), then \( 1/g(z) \) has a simple pole at \( z_0 \).

1.2. Show that \( \text{Res}(1/g, z_0) = 1/g'(z_0) \).

1.3. Let \( f(z) = 1/\sin(z) \). Find all the poles, show that they are simple, and use the previous part to find the residues at these poles.

\textbf{Ans:} See Property 5 on page 5 of Section 8 for 1.1 and 1.2. See Example 8.11 for 1.3.

2.1. Let \( p(z) \) and \( q(z) \) be analytic at \( z = z_0 \). Assume \( p(z_0) \neq 0 \) and \( q \) has a simple zero at \( z_0 \). Show that \( \text{Res}_{z=z_0}(p(z)/q(z)) = p(z_0)/q'(z_0) \).

2.2. Let \( f(z) = \cot(z) \). Find all the poles, show that they are simple, and use the previous part to find residues at these poles.

\textbf{Ans:} See Example 8.13 for 2.1 and Section 8.4.3. for 2.2.

3. By using the Taylor series of \( \cos(z) \) and \( \sin(z) \) around \( z = 0 \), compute the first few terms of the Laurent expansion of \( \cot(z) \) around \( z = 0 \).

\textbf{Ans:} See Example 8.17.

4. Suppose \( f(z) \) is analytic in the region \( A \) except for a set of isolated singularities. Suppose \( C \) is a simple closed curve in \( A \) that doesn't go through any of the singularities of \( f \) and is oriented counterclockwise.

4.1. Suppose that there is only one isolated singularity inside \( C \) at the point \( z_1 \). By using the extended version of Cauchy's theorem, show that \( \int_C \frac{f(z)dz}{z-z_1} = 2 \pi i \text{Res}(f, z_1) \).

4.2. Suppose now that there are two isolated singularities inside \( C \) at the points \( z_1 \) and \( z_2 \). Again, by using the extended version of Cauchy's theorem, show that \( \int_C \frac{f(z)dz}{z-z_1} = 2 \pi i \left( \text{Res}(f, z_1) + \text{Res}(f, z_2) \right) \).

4.3. Generalize the previous part to show that

\[ \int_C f(z)dz = 2 \pi i \sum \text{residues of } f \text{ inside } C. \]

This is Cauchy's residue theorem.
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