18.04 Practice Laplace transform, Spring 2018 Solutions

On the final exam you will be given a copy of the Laplace table posted with these problems.

Problem 1.
Do each of the following directly from the definition of Laplace transform as an integral.
(a) Compute the Laplace transform of \( f_1(t) = e^{at} \).
(b) Compute the Laplace transform of \( f_2(t) = t \).
(c) Let \( F(s) = \mathcal{L}(f_1(s)) \). Prove the s-derivative rule: \( \mathcal{L}(tf(t); s) = -F'(s) \).

(a) \( \mathcal{L}(f_1(s)) = \int_0^\infty e^{at}e^{-st} \, dt = \frac{e^{(a-s)t}}{a-s} \bigg|_0^\infty = \left\{ \begin{array}{ll} 1/(s-a) & \text{if } \text{Re}(s) > \text{Re}(a) \\ \text{divergent} & \text{if } \text{Re}(a) \leq \text{Re}(s) \end{array} \right. \)

Of course, we can analytically continue this to the region \( \mathbb{C} - \{a\} \).

(b) \( \mathcal{L}(f_2(s)) = \int_0^\infty te^{-st} \, dt = \frac{te^{-st}}{-s} - \frac{e^{-st}}{s^2} \bigg|_0^\infty = \left\{ \begin{array}{ll} 1/s^2 & \text{if } \text{Re}(s) > \text{Re}(0) \\ \text{divergent} & \text{if } \text{Re}(0) \leq \text{Re}(s) \end{array} \right. \)

Of course, we can analytically continue this to the region \( \mathbb{C} - \{0\} \).

(c) (See the topic 12 notes.)
\[
F'(s) = \frac{d}{ds} \int_0^\infty f(t)e^{-st} \, dt = \int_0^\infty -tf(t)e^{-st} \, dt = \mathcal{L}(-tf(t); s). 
\]

Problem 2.
For each of the following you can use the Laplace table if it helps.
(a) Compute the Laplace transform of \( \cosh(at) \).
(b) Compute the Laplace transform of \( f(t) = \left\{ \begin{array}{ll} 0 & \text{for } t < 5 \\ \cosh(a(t - 5)) & \text{for } t > 5 \end{array} \right. \)
(c) Compute the Laplace transform of \( f(t) = \left\{ \begin{array}{ll} \sin(t) & \text{for } 0 \leq t \leq \pi \\ 0 & \text{for } t > \pi \end{array} \right. \)
(d) Compute the Laplace transform of \( t \cos(at) \).
(e) Let \( \Gamma(z) = \mathcal{L}(t^{z-1}; s = 1) \). Show that \( \Gamma(z + 1) = z\Gamma(z) \)

(a) \( \cosh(at) = \frac{e^{at} + e^{-at}}{2} \). So,
\[
\mathcal{L}(\cosh(at); s) = \frac{1}{2} \left( \frac{1}{s-a} + \frac{1}{s+a} \right) = \frac{s}{s^2 - a^2}. 
\]

(b) The \( t \)-shift rule says \( \mathcal{L}(f(s)) = e^{-5s} \mathcal{L}(\cosh(at); s) = e^{-5s} \frac{s}{s^2 - a^2} \).

(c) We could do this directly from the integrals. Instead, notice that for \( t > 0 \)
\[
f(t) = \left\{ \begin{array}{ll} 0 & \text{for } t < \pi \\ \sin(t - \pi) & \text{for } t > \pi \end{array} \right. 
\]
So, \( \mathcal{L}(f(s)) = \mathcal{L}(\sin(t); s) + e^{-\pi s} \mathcal{L}(\sin(t); s) = \left(1 + e^{-\pi s}\right) \frac{1}{s^2 + 1} \).

**Problem 3.**

(a) *Use the Laplace transform to solve the differential equation \( x' + x = te^{2t} \), with \( x(0) = 3. \)

Find the Laplace inverse using the formula involving the sums of residues. (Be sure to verify that the hypotheses of the theorem hold.)

First, the Laplace transform of \( te^{2t} = \frac{d}{ds} \left( \frac{1}{s-2} \right) = \frac{1}{(s-2)^2} \).

Now, take the Laplace transform of the differential equation

\[
(sX(s) - x(0)) + X(s) = \frac{1}{(s-2)^2} \implies X(s) = \frac{1}{(s+1)(s-2)^2} + \frac{3}{x+1}.
\]

Since \( X(s) \) decays faster than \( 1/s \) we can apply this inversion formula using the sums of the residues.

\[
x(t) = \sum \text{residues of } e^{st}X(s).
\]

We split \( e^{st}X(s) \) into two pieces \( F_1(s) = \frac{e^{st}}{(s-2)^2(s+1)} \) and \( F_2(s) = \frac{3}{s+1} \). \( F_1 \) has poles at \( s = 2 \) and \( s = -1 \).

At \( s = 2 \): Let \( G(s) = (s-2)^2F_1(s) = \frac{e^{st}}{s+1} \). \( G(s) \) is analytic at \( s = 2 \), so \( G(s) = G(2) + G'(2)(s-2) + \ldots \). Thus, \( \text{Res}(F_1,2) = G'(2) = \frac{te^{2t} - e^{2t}}{3} = \frac{te^{2t}}{3} - \frac{e^{2t}}{9} \).

At \( s = -1 \): \( \text{Res}(F_1,-1) = \lim_{s \to -1} (s+1)F_1(s) = \frac{e^{-t}}{9} \).

So, \( \mathcal{L}^{-1}(F_1; t) = \text{Res}(F_1,2) + \text{Res}(F_1,-1) = \frac{te^{2t}}{3} - \frac{e^{2t}}{9} + \frac{e^{-t}}{9} \).

Clearly, \( \mathcal{L}^{-1}(F_2; t) = 3e^{-t} \). Thus, \( x(t) = \frac{te^{2t}}{3} - \frac{e^{2t}}{9} + \frac{e^{-t}}{9} + 3e^{-t} \).

(b) *Solve \( y' - y = \begin{cases} 0 & \text{for } t < 1 \\ 1 & \text{for } t > 1 \end{cases} \), with \( y(0) = 0. \)
Why does the inversion formula involving sums of residues not apply?

Let \( f(t) = \begin{cases} 0 & \text{for } t < 1 \\ 1 & \text{for } t > 1 \end{cases} \). Since \( \mathcal{L}(1; s) = 1/s \), the \( t \)-shift rule shows that \( \mathcal{L}(f; s) = e^{-s}/s \).

Now take the Laplace transform of the differential equation:

\[
(s - 1)Y(s) = \frac{e^{-s}}{s} \Rightarrow Y(s) = \frac{e^{-s}}{s(s - 1)}
\]

Using partial fractions we find that

\[
\frac{1}{s(s - 1)} = -\frac{1}{s} + \frac{1}{s - 1}, \quad \text{so} \quad \mathcal{L}^{-1} \left( \frac{1}{s(s - 1)}; t \right) = -1 + e^t.
\]

Now using the \( t \)-shift rule we have

\[
y(t) = \begin{cases} 0 & \text{for } t < 1 \\ -1 + e^{t-1} & \text{for } t > 1. \end{cases}
\]

Problem 4.

Use the Laplace transform to solve the differential equation \( x'' + x = \sin(t) \), with \( x(0) = 0, x'(0) = 0 \).

(Hint: use the table to do the Laplace inverse.)

The zero initial conditions make taking the Laplace transform of the differential equation easy

\[
(s^2 + 1)X(s) = \frac{1}{s^2 + 1} \Rightarrow X(s) = \frac{1}{(s^2 + 1)^2}.
\]

This is in our Laplace table. So, \[
x(t) = \frac{1}{2} \left( \sin(t) - t \cos(t) \right).
\]