

## 18.05 Problem Set 4, Spring 2014

**Problem 1.** (10 pts.) **Time to failure.** Recall that an exponential random variable  $X \sim \text{exp}(\lambda)$  has mean  $\frac{1}{\lambda}$  and pdf given by  $f(x) = \lambda e^{-\lambda x}$  on  $x \geq 0$ .

(a) Compute  $P(X \geq x)$ .

(b) Suppose that  $X_1$  and  $X_2$  are independent exponential random variables with mean  $\frac{1}{\lambda}$ . Let  $T = \min(X_1, X_2)$ . Find the cdf of  $T$ . (Hint: what is  $P(T \geq t)$ ?)

(c) Suppose we are testing 3 different brands of light bulbs  $B_1$ ,  $B_2$ , and  $B_3$  whose lifetimes are exponential random variables with mean  $1/2$ ,  $1/3$ , and  $1/5$  years, respectively. Assuming that all of the bulbs are independent, what is the expected time before one of the bulb fails. (Hint: part (b) was a warmup for this problem.)

**Problem 2.** (10 pts.) **Aching joints.** Suppose  $X$  and  $Y$  have joint pdf  $f(x, y) = c(x^2 + xy)$  on  $[0, 1] \times [0, 1]$ .

(a) Find  $c$  and the joint cdf  $F(x, y)$ .

(b) Find the marginal cumulative distribution functions  $F_X$  and  $F_Y$  and the marginal pdf  $f_X$  and  $f_Y$ .

(c) Find  $E(X)$  and  $\text{Var}(X)$ .

(d) Find the covariance and correlation of  $X$  and  $Y$ .

**Problem 3.** (10 pts.) **Elections.** To head the newly formed US Dept. of Statistics, suppose that 50% of the population supports Erika, 20% supports Ruthi, and the rest are split between Peter, Jon and Jerry. A poll asks 400 random people who they support.

(a) Use the central limit theorem to estimate the probability that at least 52.5% of those polled prefer Erika?

(b) Use the central limit theorem to estimate the probability that less than 25% of those polled prefer Peter, Jon or Jerry?

**Problem 4.** (10 pts.) **Canada has no common cents.** To save a mint, in 2012 Canada decided to do away with its pennies. The Chubby Chef in Equality, Illinois wants to be ready should the United States decide to pass a similar law. The Chubby Chef processes  $n = 1000$  orders of assorted baked goods each day, and will round the price of each order to the nearest nickel (e.g., \$3.57 rounds to \$3.55 while \$3.58 rounds to \$3.60). Let  $p$  be the probability that the total rounding error over the course of a day exceeds  $t = 100$  cents (in absolute value).

Estimate  $p$  using the central limit theorem.

**Extra credit 5 points** Simulate this in R with 10000 trials. (Each trial involves 1000 orders.) Print out or hand copy your code and include it. Give the result of running your code 3 times,

**Problem 5.** (10 pts.) **Independence.** Suppose  $X$  and  $Y$  are random variables with the following joint pmf. Are  $X$  and  $Y$  independent?

$X \setminus Y$	1	2	3
1	1/18	1/9	1/6
2	1/9	1/6	1/18
3	1/6	1/18	1/9

**Problem 6.** (10 pts.) **Correlation.** Suppose  $X$  and  $Y$  are random variables with  $P(X = 1) = P(X = -1) = \frac{1}{2}$ ;  $P(Y = 1) = P(Y = -1) = \frac{1}{2}$ . Let  $c = P(X = 1 \text{ and } Y = 1)$ .

- (a) Determine the joint distribution of  $X$  and  $Y$ ,  $\text{Cov}(X, Y)$ , and  $\text{Cor}(X, Y)$ .
- (b) For what value(s) of  $c$  are  $X$  and  $Y$  independent? For what value(s) of  $c$  are  $X$  and  $Y$  100% correlated?

**Problem 7.** (10 pts.) **Don't be late!** Alice and Bob are trying to meet for lunch and both will arrive, independently of each other, uniformly and at random between noon and 1pm. Let  $A$  and  $B$  be the number of minutes after noon at which Alice and Bob arrive, respectively. Then  $A$  and  $B$  are independent uniformly distributed random variables on  $[0, 60]$ .

**Hint:** For parts (c-e) you might find it easiest to find the fraction of the square  $[0, 60] \times [0, 60]$  filled by the event.

- (a) Find the joint pdf  $f(a, b)$  and joint cdf  $F(a, b)$ .
- (b) Find the probability that Alice arrives before 12:30.
- (c) Find the probability that Alice arrives before 12:15 and Bob arrives between 12:30 and 12:45 in two ways:
- By using the fact that  $A$  and  $B$  are independent.
  - By shading the corresponding area of the square  $[0, 60] \times [0, 60]$  and showing that the proportion of shaded area equals the probability.
- (d) Find the probability that Alice arrives less than five minutes after Bob. (Hint: use method (ii) from part (c).)
- (e) Now suppose that Alice and Bob are both rather impatient and will leave if they have to wait more than 15 minutes for the other to arrive. What is the probability that Alice and Bob will have lunch together?

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